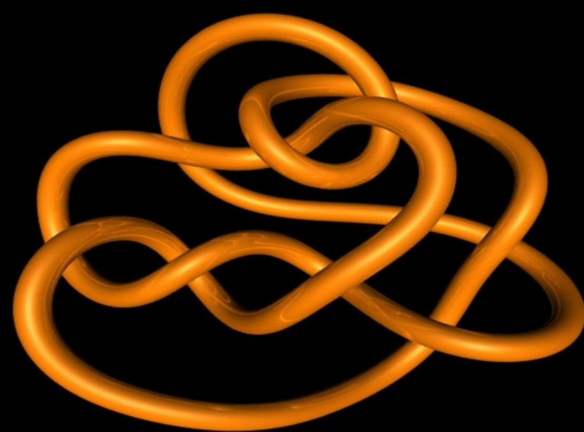
Arcs, knots, and convex embeddability

(joint with Iannella, Kulikov, Marcone)

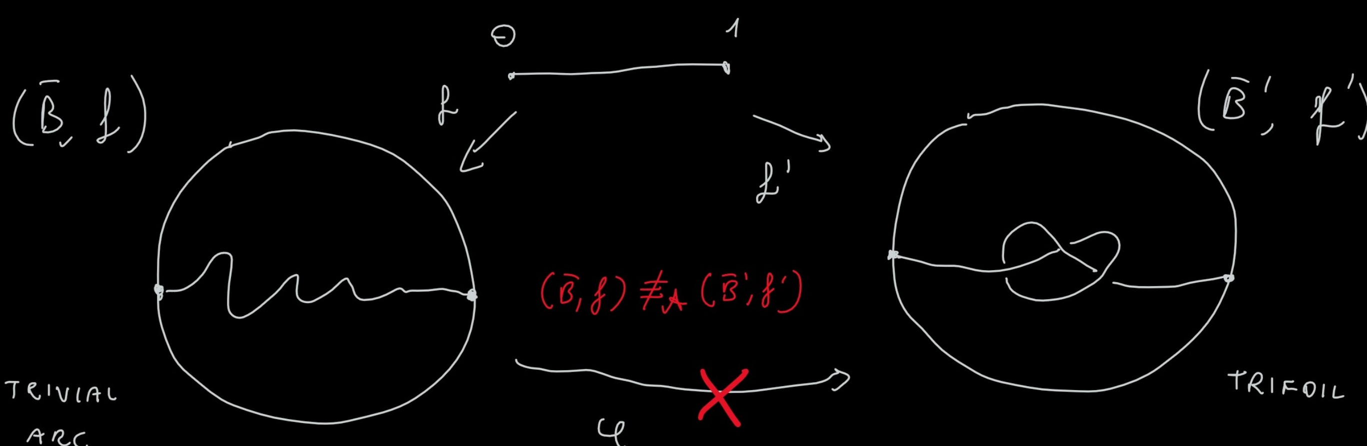


Part 1: Introduction   Part 2: Arcs and linear orders   Part 3: Knots and circular orders

Definition

A (proper) **arc** is a pair $(\bar{B}, f)$ with $\bar{B}$ a closed ball in $\mathbb{R}^3$ and $f\colon [0;1] \to \bar{B}$ a topological embedding with $f(x) \in \partial\bar{B} \iff x = 0 \lor x = 1$. The collection of all arcs is denoted by $\mathcal{A}$.

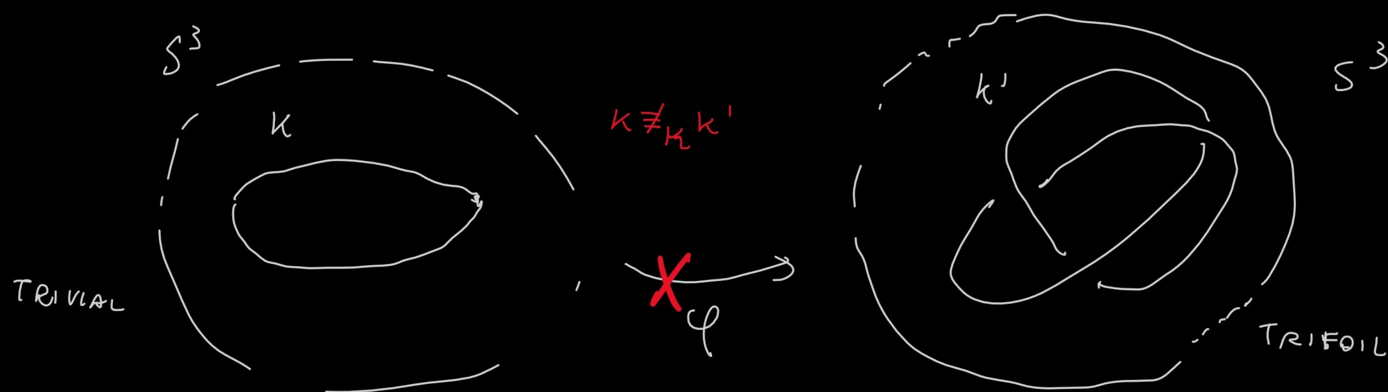
Two arcs $(\bar{B}, f)$ and $(\bar{B}', f')$ are **equivalent**, in symbols $(\bar{B}, f) \equiv_{\mathcal{A}} (\bar{B}', f')$, if there is a homeomorphism $\varphi\colon \bar{B} \to \bar{B}'$ such that $\varphi(f) = f'$.



Definition

A **knot** is an embedding of the circle S^1 into the 3-sphere S^3. The collection of all knots is denoted by $\mathcal{K}$.

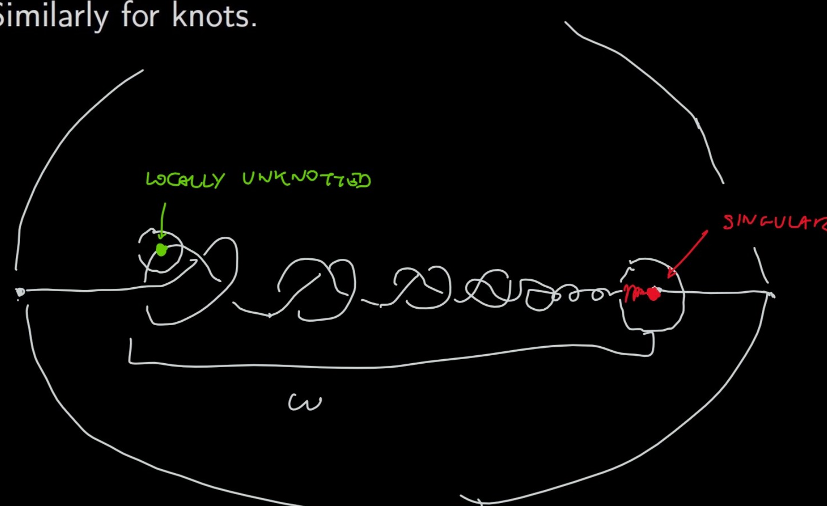
Two knots $K, K' \in \mathcal{K}$ are **equivalent**, in symbols $K \equiv_{\mathcal{K}} K'$, if there is a homeomorphism $\varphi\colon S^3 \to S^3$ such that $\varphi(K) = K'$.



Definition

A point $x \in f$ of an arc $(\bar{B}, f) \in \mathcal{A}$ is called **singular point** if $(\bar{B}', f \cap \bar{B}')$ is not trivial for every $x \in \bar{B}' \subseteq \bar{B}$ (with $x \notin \partial\bar{B}'$ if x is not an endpoint of f) for which $(\bar{B}', f \cap \bar{B}') \in \mathcal{A}$.

The arc $(\bar{B}, f)$ is **tame** (or **locally unknotted**) if it has no singular points, and **wild** otherwise. Similarly for knots.



Definition

Let R, S be binary relations on standard Borel spaces X, Y. We write $R \leq_B S$ if there is a Borel function $f\colon X \to Y$ such that for all $x, x' \in X$

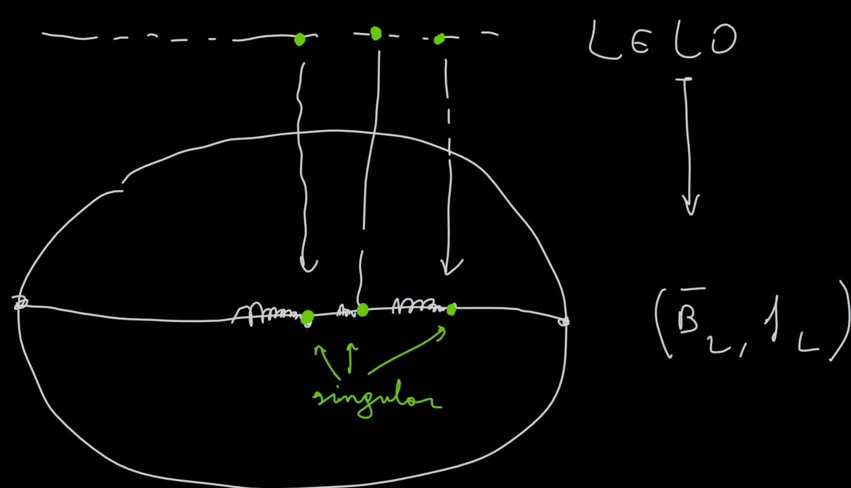
$$x \; R \; x' \iff f(x) \; S \; f(y).$$

We write $R \sim_B S$ for $R \leq_B S \leq_B R$.

THM (Kulikov, 2017): $\cong_{LO} \not\leq_B \equiv_{\mathcal{A}}, \equiv_K$

Isomorphism on ctbl linear orders (it is S_∞-complete).

Dim: (1) $\cong_{LO} \leq_B \equiv_{\mathcal{A}}$

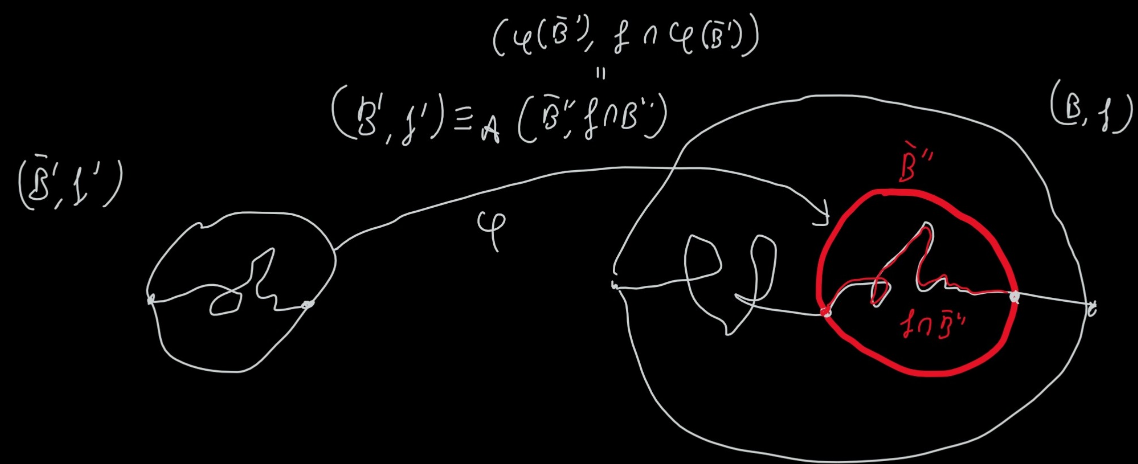


(2) $\equiv_{\mathcal{A}} \not\leq_B \cong_{LO}$: there is a turbulent $G \curvearrowright X$ s.t. $E_G \leq_B \equiv_{\mathcal{A}}$

Definition

An arc $(\bar{B}', f')$ is a **component** of the arc $(\bar{B}, f)$, in symbols $(\bar{B}', f') \precsim_{\mathcal{A}} (\bar{B}, f)$, if there is a topological embedding $\varphi\colon \bar{B}' \to \bar{B}$ such that $\varphi(f') = f \cap \operatorname{rng}(\varphi)$.

The induced equivalence relation $\approx_{\mathcal{A}}$ is called **mutual component** relation.



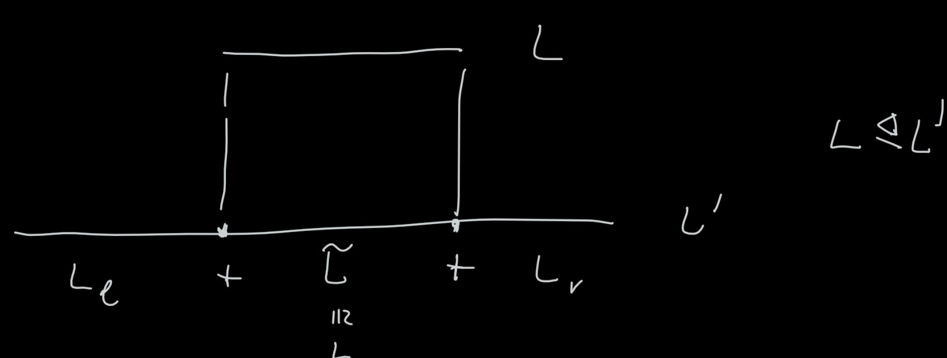
Definition

The relation $\trianglelefteq_{\mathrm{LO}}$ of **convex embeddability** on LO is defined by

$$L \trianglelefteq_{\mathrm{LO}} L' \iff L' = L_l + \tilde{L} + L_r$$

with $\tilde{L} \cong L$ (the linear orders L_l, L_r might be empty, finite, or infinite).

The induced equivalence relation $\bowtie_{\mathrm{LO}}$ is called **convex biembeddability**.



Rmk: $L \cong L' \Rightarrow L \bowtie L'$ and $L \trianglelefteq L' \Rightarrow L \sqsubseteq L'$

THM: $\trianglelefteq_{LO} \leq_B \lesssim_A$, $\bowtie_{LO} \leq_B \approx_A$

COMBINATORIAL PROPERTIES:

(1) $(\mathrm{Int}_{\mathbb{R}}, \subseteq) \leq_B (LO, \trianglelefteq_{LO})$.

↳ open intervals

Fix a bijection $f: \mathbb{Q} \to \omega \setminus \{0,1\}$: send $\underset{\mathrm{Int}_{\mathbb{R}}}{(a;b)} \mapsto \sum_{q \in \mathbb{Q} \cap (a;b)} f(q)$.

Cor: $\trianglelefteq_{LO}$ has antichains and chains of size $2^{\aleph_0}$.

(2) On WO (well-orders), $\trianglelefteq_{WO} = \sqsubseteq_{WO}$ and $\cong_{WO} = \bowtie_{WO}$: but WO is unbounded in LO w.r.t. $\trianglelefteq_{LO}$.

(3) $(LO, \trianglelefteq_{LO})$ embeds into any upper cone $(\nabla_{\bar{L}}, \trianglelefteq_{LO})$ where $\nabla_{\bar{L}} = \{ L \in LO \mid \bar{L} \trianglelefteq L \}$.

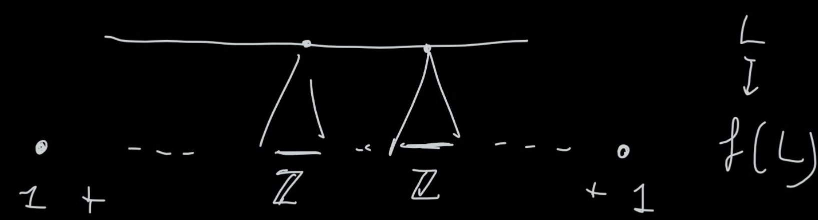
(L) $\mathcal{F}$ is a dominating family if $\forall L\ \exists L' \in \mathcal{F}\ \ L \trianglelefteq L'$

$\mathcal{B}$ is a basis if $\forall L\ \exists L' \in \mathcal{B}\ \ L' \trianglelefteq L$

all dominating families and all basis have maximal size $2^{\aleph_0}$.

Prop: $\cong_{LO} \leq_B \bowtie_{LO}$

Pf: $LO \ni L \longmapsto 1 + \overbrace{\mathbb{Z} \times L}^{\text{antilexicographical order}} + 1 = f(L)$



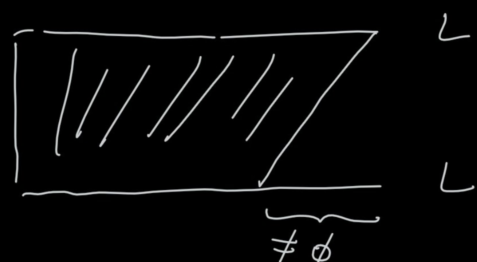
Definition

A linear order L is called

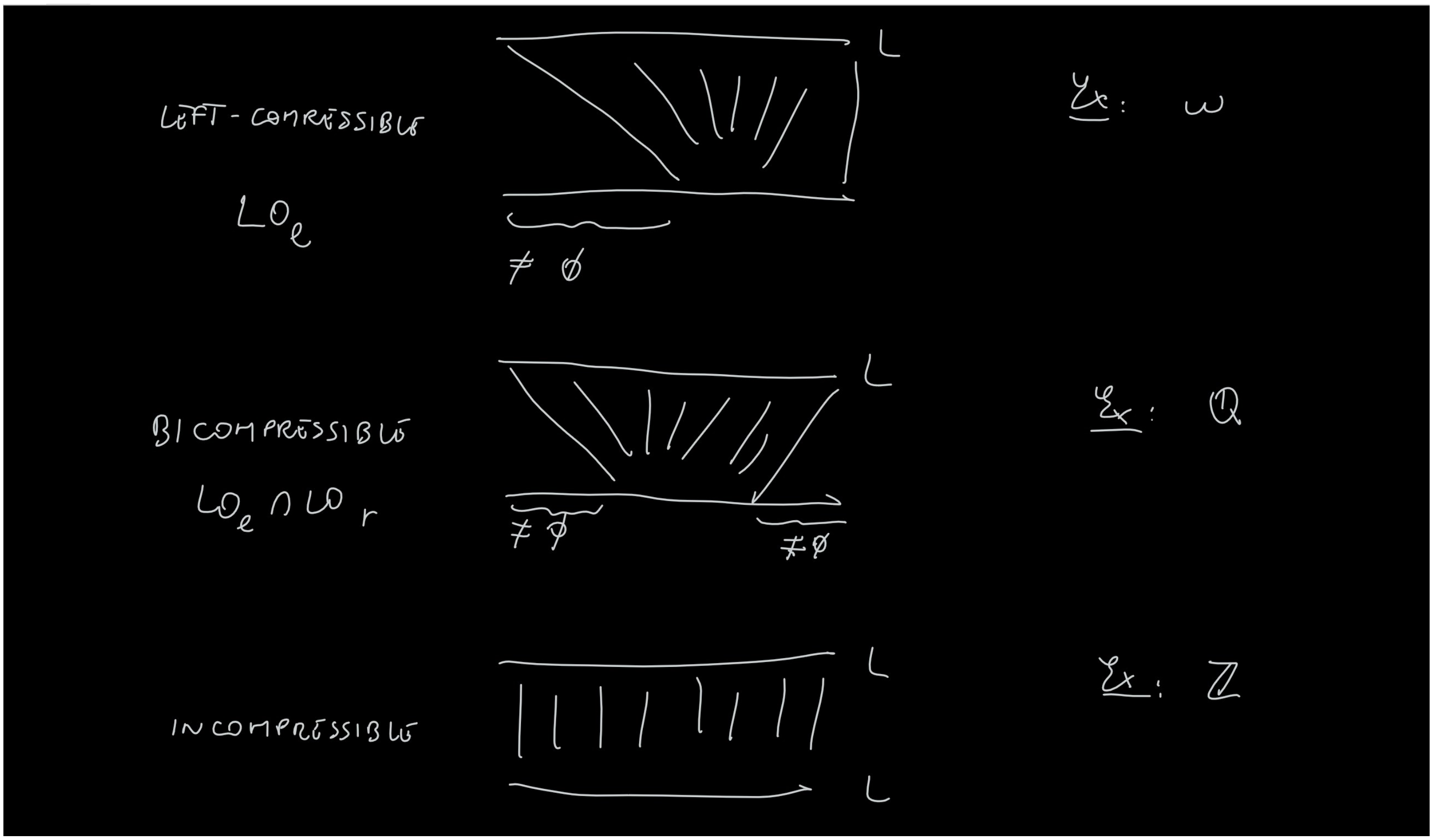
- **right compressible** if $L = \tilde{L} + L_r$ with $L_r \neq \emptyset$ and $\tilde{L} \cong L$
- **left compressible** if $L = L_l + \tilde{L}$ with $L_l \neq \emptyset$ and $\tilde{L} \cong L$
- **bicompressible** if $L = L_l + \tilde{L} + L_r$ with $L_l, L_r \neq \emptyset$ and $\tilde{L} \cong L$ (equivalently, L is both right and left compressible)
- **incompressible** if it is neither right nor left compressible

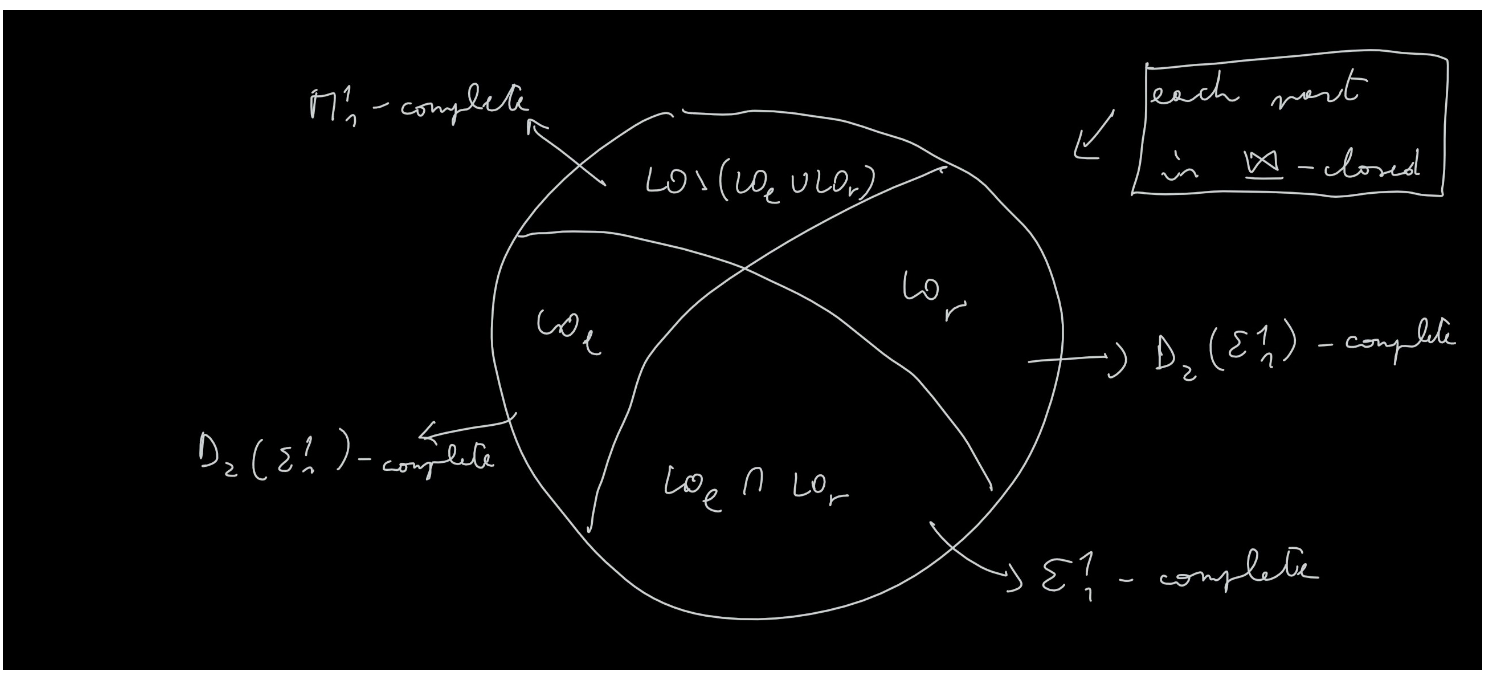
RIGHT-COMPRESSIBLE

LO_r



Ex: ω^*


LEFT-COMPRESSIBLE
LO_ℓ
L
$\neq \emptyset$
Ex: ω
BICOMPRESSIBLE
$LO_\ell \cap LO_r$
L
$\neq \emptyset$
$\neq \emptyset$
Ex: $\mathbb{Q}$
INCOMPRESSIBLE
L
L
Ex: $\mathbb{Z}$


Π^1_1-complete
$LO \setminus (LO_\ell \cup LO_r)$
each part in $\bowtie$-closed
LO_ℓ
LO_r
$D_2(\Sigma^1_1)$-complete
$D_2(\Sigma^1_1)$-complete
$LO_\ell \cap LO_r$
Σ^1_1-complete

THM: On each of the 4 parts, $\bowtie_{LO}$ is Borel bi-reducible with $\cong_{LO}$.

Pf: (1) The maps

$$\left.\begin{array}{lll} L \mapsto 1 + \mathbb{Z} \times L + 1 & \in & LO \setminus (LO_\ell \cup LO_r) \\ L \mapsto 1 + \mathbb{Z} \times L + \mathbb{Q} & \in & LO_r \setminus LO_\ell \\ L \mapsto \mathbb{Q} + \mathbb{Z} \times L + 1 & \in & LO_\ell \setminus LO_r \\ L \mapsto \mathbb{Q} + \mathbb{Z} \times L + \mathbb{Q} & \in & LO_\ell \cap LO_r \end{array}\right]$$

reduce $\cong_{LO}$ to $\bowtie_{LO}$ restricted to the corresponding part.

(2) On $LO \setminus (LO_\ell \cup LO_r)$, the relations $\cong_{LO}$ and $\bowtie_{LO}$ coincide.

(3) On the remaining three parts, we reduce $\bowtie_{LO}$ to $\cong_{LO}^+ \sim_B \cong_{LO}$, where the FS-jump $E \mapsto E^+$ is defined by

Definition

The **Friedman-Stanley jump** E^+ of an equivalence relation E on X is the relation on X^ω defined by

$$(x_n)_{n\in\omega}\, E^+\, (y_n)_{n\in\omega} \iff \{[x_n]_E \mid n \in \omega\} = \{[y_n]_E \mid n \in \omega\}.$$

For example on $LO_\ell \curvearrowright LO_r$ the reduction is

$$L \longmapsto \{ [n;m]_L \mid n <_L m,\ [n;m]_L \text{ is infinite} \}.$$

$\square$

<u>COR</u>: If $G \curvearrowright X$ is turbulent, then $E_G \not\leq_B \bowtie_{LO}$.

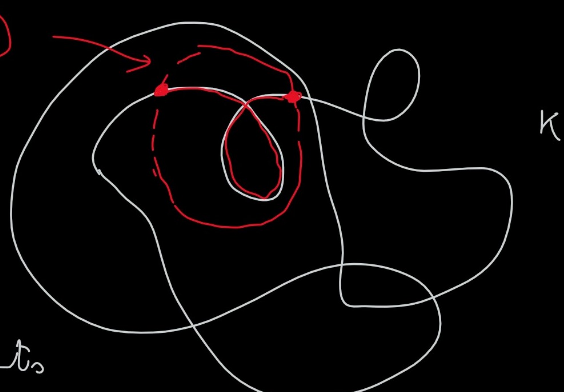
<u>THM</u>: $\equiv_{\mathcal{A}} \leq_B \approx_{\mathcal{A}}$

<u>COR</u>: $\trianglelefteq_{LO} <_B \precsim_{\mathcal{A}}$ and $\bowtie_{LO} <_B \approx_{\mathcal{A}}$.

<u>Definition</u>

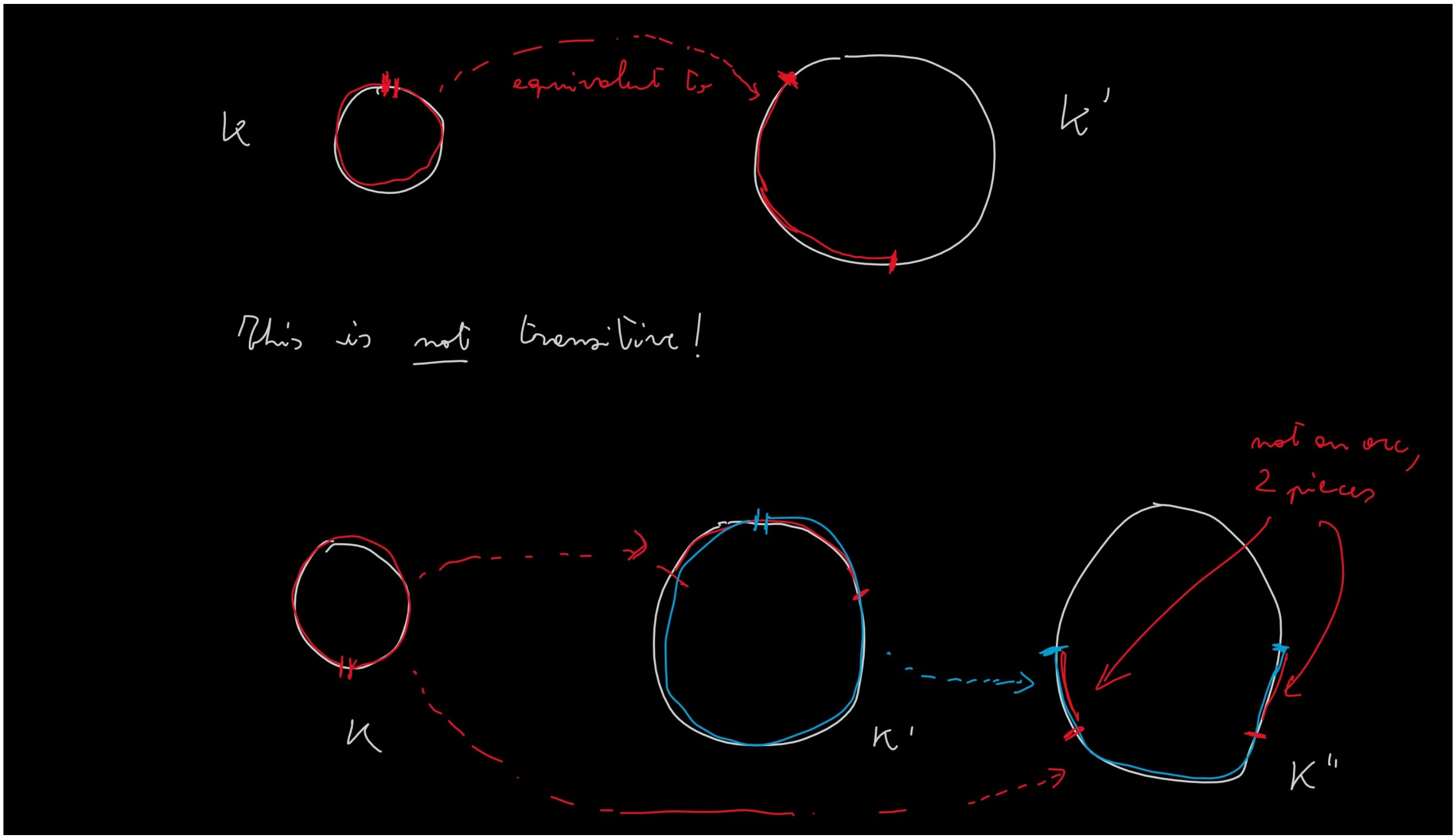
A **sub-arc** of a knot K is a pair $(\bar{B}, K \cap \bar{B})$ such that $\partial \bar{B}$ meets transversely K in exactly two points and $(\bar{B}, K \cap \bar{B})$ is an arc.

sub-arc of K $\rightarrow$ $(\bar{B}, K \cap \bar{B})$



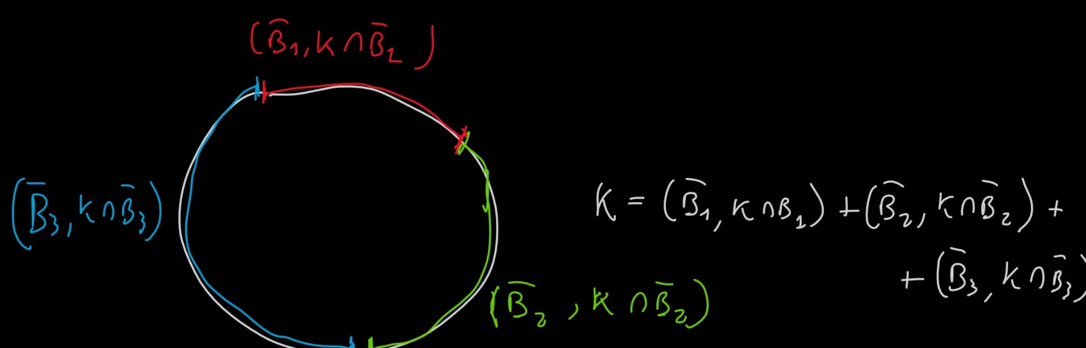
becomes a knot if we glue its endpoints

<u>Rmk</u>: This seems to give a "sub-knot relation": $K \precsim_K K'$



Definition

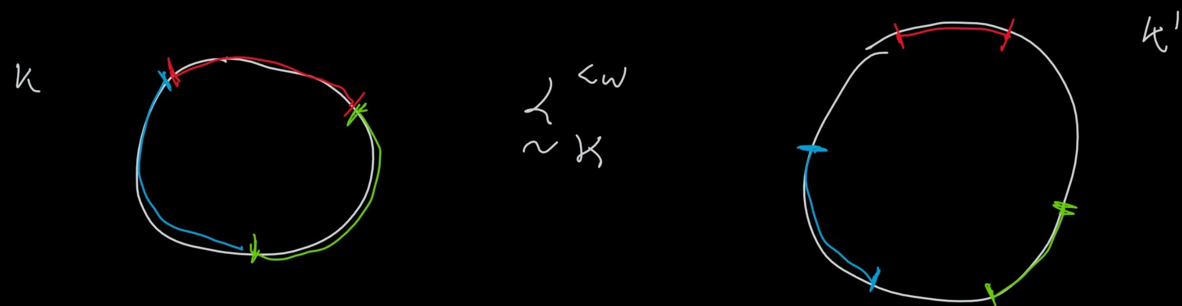
The (oriented) knot K is a **(finite) tame sum** of the (oriented) sub-arcs $(\bar{B}_1, K \cap \bar{B}_1), \ldots, (\bar{B}_n, K \cap \bar{B}_n)$ if the $\bar{B}_i$'s are ordered according to the orientation of K, almost pairwise disjoint (they meet in at most one point, and this can happen only if they are consecutive), and K becomes tame if we substitute each $(\bar{B}_i, K \cap \bar{B}_i)$ with a trivial arc with same ambient sphere $\bar{B}_i$ and same endpoints.



Definition

A knot K is a (**finite**) **piecewise component** of K', in symbols $K \precsim_{\mathcal{K}}^{<\omega} K'$, if K is a finite tame sum of sub-arcs, each of which is equivalent to a corresponding sub-arc of K' (of course we require such sub-arcs to be ordered in the same way by an orientation of K' and almost pairwise disjoint).

The induced equivalence relation $\approx_{\mathcal{K}}^{<\omega}$ is called **mutual** (**finite**) **piecewise component** relation.



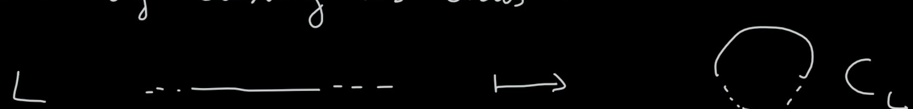
We now pull-back this to linear orders: actually, it is more natural to work with circular orders.

Definition

A ternary relation $C \subseteq X^3$ on a set X is said to be a **circular order** if the following conditions are satisfied:

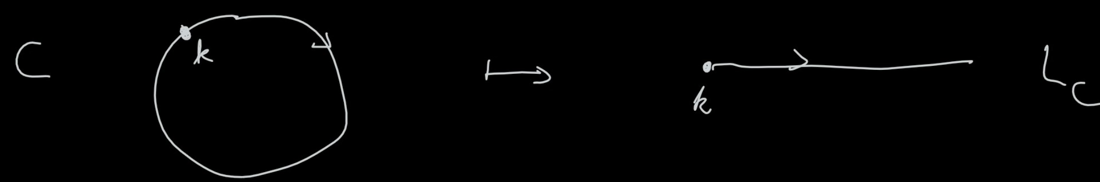
- $(x, y, z) \in C \Rightarrow (y, z, x) \in C$ *(cyclicity)*
- $(x, y, z) \in C \Rightarrow (y, x, z) \notin C$ *(asymmetry)*
- $(x, y, z), (x, z, w) \in C \Rightarrow (x, y, w) \in C$ *(transitivity)*
- if $x, y, z \in X$ are distinct, then $(x, y, z) \in C$ or $(x, z, y) \in C$ *(totality)*

A linear order can be transformed into a circular order by "closing its ends":



This is <u>not</u> a reduction of $\cong_{LO}$ to $\cong_{CO}$ (problem: rotations!)

Conversely, given $C \in \mathrm{CO}$ and a point $k \in C$ we can get a linear order L_C with k as a minimum



There are $|C|$-many choices (and many variants, e.g. we can put k as a maximum).

<u>THM</u>: $\cong_{\mathrm{CO}} \sim_B \cong_{\mathrm{LO}}$

<u>THM</u>: $\sqsubseteq_{\mathrm{CO}}$ (embeddability on circular orders) is a wqo.

As for convex embeddability:

<u>Definition</u>
A subset A of a circular order C is **convex** if for all $x, y \in A$ either $z \in A$ for all z such that $(x, z, y) \in C$, or $z \in A$ for all z such that $(y, z, x) \in C$.

<u>Definition</u> (flawed)
The relation $\trianglelefteq_{\mathrm{CO}}$ of **convex embeddability** on CO is defined by

$$C \trianglelefteq_{\mathrm{CO}} C' \iff C' = \tilde{C} + C^{(r)}$$

with $\tilde{C} \cong C$ (again $C^{(r)}$ might be empty, finite, or infinite).

<u>Rmk</u>: Same problem as for knots: $\trianglelefteq_{\mathrm{CO}}$ is <u>not</u> transitive.

Definition

A circular order C is a (**finite**) **sum** of its convex subsets $C_1, \ldots, C_n$, in symbols $C = \sum_{i=1}^n C_i$, if the C_i's are pairwise distinct, $C = \bigcup_{1 \le i \le n} C_i$ and $(x, y, z) \in C$ for all $x \in C_i$, $y \in C_j$, $z \in C_k$ with $i < j < k$.

Definition

The relation $\trianglelefteq_{\mathrm{CO}}^{<\omega}$ of (**finite**) **piecewise convex embeddability** on CO is defined by

$$C \trianglelefteq_{\mathrm{CO}}^{<\omega} C' \iff C = \sum_{i=1}^{n} C_i \text{ (for some } n \in \omega) \text{ and } C' = \sum_{i=1}^{n} (\tilde{C}_i + C_i^{(r)})$$

with $\tilde{C}_i \cong C_i$ (each $C_i^{(r)}$ might be empty, finite, or infinite).

The induced equivalence relation $\bowtie_{\mathrm{CO}}^{<\omega}$ is called (**finite**) **piecewise convex biembeddability**.

THM: $\trianglelefteq_{CO}^{<\omega} \le_B \precsim_K^{<\omega}$, hence $\bowtie_{CO}^{<\omega} \le_B \approx_K^{<\omega}$.

THM: $\cong_{LO} \le_B \bowtie_{CO}^{<\omega}$

Given $(x_n)_{n\in\omega}, (y_n)_{n\in\omega} \in \mathbb{R}^\omega$, set $(x_n)_{n\in\omega} \, E_1 \, (y_n)_{n\in\omega}$ if $\exists n \, \forall m \ge n \, (x_m = y_m)$. E_1 is not reducible to an orbit equivalence relation

THM: $E_1 \le_B \bowtie_{CO}^{<\omega}$

COR: $\bowtie_{CO}^{<\omega}$ is not reducible to an orbit eq. rel.

COR: $\cong_{LO} <_B \bowtie_{CO}^{<\omega}$