Schreier decorations of unimodular random graphs

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Schreier graphs

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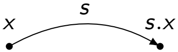
Vertices: $x \in X$. Edges: 

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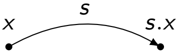
Vertices: $x \in X$. Edges: 

Notation: $\mathrm{Sch}(\Gamma, S, X)$. Oriented, labeled graph.

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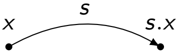
Vertices: $x \in X$. Edges: 

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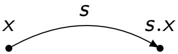
Vertices: $x \in X$. Edges: 

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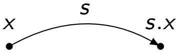
Vertices: $x \in X$. Edges: 

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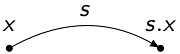
Vertices: $x \in X$. Edges: 

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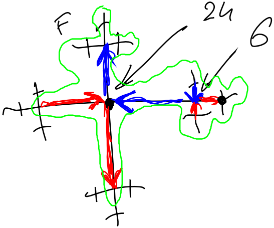
Folklore combinatorics: every $2d$-regular finite graph is a Schreier graph of the free group F_d.

I.e. the edges have an orientation and d-coloring such that at each vertex there is exactly one incoming and outgoing edge of each color. (We call this a Schreier decoration.)

Task: Find an $\mathrm{Aut}(T_4)$-invariant random Schreier decoration of T_4.



Start from root, go radially outwards.

Claim: does not depend on the choice of root in distr.

Proof: $\mathbb{P}[\text{seeing a fixed dec in } F]$

$$= \frac{1}{24} \cdot \left(\frac{1}{6}\right)^{|F|-1}$$

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- Invariant random rooted Schreier graph $=$ coset Schreier graph of an IRS.
- Grebik has a quasi-p.m.p. version now.

Definition (graphing)

Let (X,μ) be a standard Borel probability space. A graphing is a locally finite graph $\mathcal{G}$ on $V(\mathcal{G}) = X$ with Borel edge set $E(\mathcal{G}) \subset X \times X$ satisfying

$$\int_A \deg_B(x)\ d\mu(x) = \int_B \deg_A(x)\ d\mu(x)$$

for all measurable sets $A, B \subseteq X$, where $\deg_S(x)$ is the number of edges from $x \in X$ to $S \subseteq X$.

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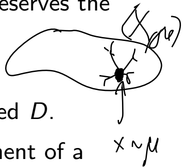

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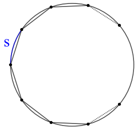
Intrinsic formulation of the p.m.p. condition: involution invariance, or Mass Transport Principle, or reversibility of random walk.

Examples

Boring example: s irrational rotation of the circle. $\mathcal{G}$: connect every point to its rotated image.



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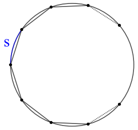

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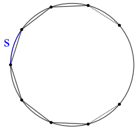

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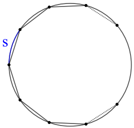

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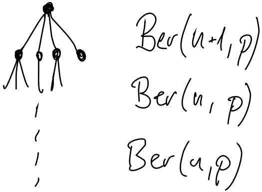
Less boring example: Galton-Watson tree.



T_{n+1}

$\{(T_{n+1}, o, \ell) \mid \ell : V(T_n) \to [0,1]$

$\rightsquigarrow$ Bernoulli graphing.

Invariant random (rooted) Schreier graphs

Random rooted Schreier graph: $(\tilde{G}, \tilde{o}, \mathrm{dec})$ random, where dec is the combinatorial data. I.e. $\tilde{\nu} \in P(\mathfrak{G}_\bullet^{\mathrm{Sch}})$, where $\mathfrak{G}_\bullet^{\mathrm{Sch}}$ is the space of rooted Schreier graphs.

Invariance: F_d acts on rooted Schreier graphs by moving the root: $s.(\tilde{G}, \tilde{o}, \mathrm{dec}) = (\tilde{G}, s.\tilde{o}, \mathrm{dec})$. Want $F_d \curvearrowright (\mathfrak{G}_\bullet^{\mathrm{Sch}}, \tilde{\nu})$ to be p.m.p.

Two viewpoints

Random rooted graphs	Measurable combinatorics
Unimodular random rooted graph $(G, o) \sim \nu$ where $\nu \in P(\mathfrak{G}_\bullet^D)$ unimod.	Graphing $\mathcal{G} = (X, E, \mu)$

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Back to the example

η the unique $\mathrm{Aut}(T_4)$-inv random Sch dec.

$\mathrm{Sch}(T_4) = \{$ Sch F_2-decorations of $T_4\}$

<u>Question</u>: Is η a factor of iid?

$\Psi : [0,1]^{V(T_4)} \to \mathrm{Sch}(T_4)$ s.t.

- Ψ is $\mathrm{Aut}(T_4)$-equiv.
- $\Psi_* u^{V(T_4)} = \eta$ where u is unif.

<u>Ex</u>: FIID independent subset of $V(T_4)$ on $[0,1]$. of density $1/5$

$\ell(v) \in [0,1]$ which is uniform random indep across V

$v \in I$ iff $\ell(u) < \ell(v)$ $\forall u \in N(v)$

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Theorem (Bencs, Hrušková, T., 2021)

The d-dimensional Euclidean grid has a factor of iid Schreier decoration for all $d \geq 2$. Same for all Archimedean lattices of even degree in the plane.

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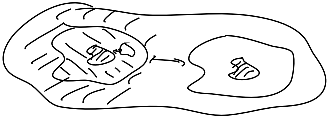
The d-dimensional Euclidean grid has a factor of iid Schreier decoration for all $d \geq 2$. Same for all Archimedean lattices of even degree in the plane.

Builds on toasts, could have Borel version.



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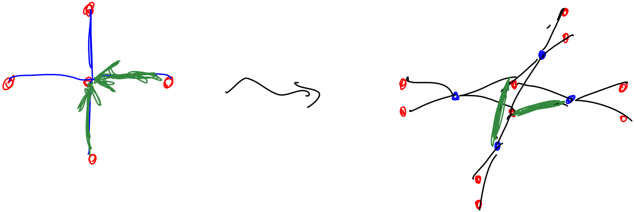

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- Not true for unimodular random rooted graphs in general.

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- Is measurably decorating the Bernoulli graphing the most difficult among graphings with the same local statistics?

Thank you for your attention

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