Set theory and a proposed model of the mind in psychology

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Joint work with Jens Mammen (U. of Aalborg)

Jens Mammen's theory

The Danish psychologist Jens Mammen has proposed a general theory for what may be called the "interface" between the inner world of a human mind (such as your own), and the outer world which this human lives in, perceives, and interacts with, and reflects upon.

From a mathematical point of view, Mammen's theory is unusual and interesting because it is formulated mathematically: Mammen formulates his theory axiomatically, describing it in terms of:

- A set U of objects in the world (the "universe").
- A topology $\mathcal{S}$ on U.
- A collection $\mathcal{C}$ of subsets of U.
- $\mathcal{S}$ and $\mathcal{C}$ must satisfy certain axioms (to be stated in a moment).

Remark: Mammen's theory is "extensional" or "extrinsic". How the brain internally realizes the model is *not* addressed.

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Mammen's theory: Why?

Some features and motivations (roughly):

- Historically, theoretical psychologists and philosophers have proposed models for the mind-world interface using broad "categories" that the mind supposedly organizes data about the world into.
  - For instance, a mind that has "experienced", or **sensed**, one or more stones is thought to have formed a broad category of "stones", and can then recognize when an object is a stone.
- What these models are missing is our relationship to individual objects/people/animals/etc: Why, if I drop a particular stone from my hand, can I identify that it is the same stone that is now on the floor as the one I had in my hand earlier?
- In psychology, this gap is particularly problematic, since attachments to **individual people** is at the core of human psychology:
  - *My* father may belong to the broad category of fathers, but *my* father's importance to *me* is not primarily derived from that he belongs to the general category of people who are fathers.

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Mammen's theory: Remark

You may find it natural to ask:

> *Could the problem of individual identification not be solved as follows: An object (or a person) is identified because it (or he or she) is uniquely determined by its membership in a number of categories?*

Perhaps, but this still does not seem to reflect accurately psychologists' observations "in the field" (e.g., the clinical setting) of how the human mind works, especially when it comes to people.

From the working psychologist's point of view, it seems that our mind doesn't just form broad "sense" categories, but also places individual people and objects in special **choice**, or "distinguished" categories, especially when it comes to psychological attachments.

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Mathematical formulation

Definition

A "Mammen model" is a non-empty set U equipped with:

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  - There is a non-empty $C \in \mathcal{C}$.
  - Every non-empty $C \in \mathcal{C}$ contains a singleton which is in $\mathcal{C}$.
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- Further, $\mathcal{S}$ and $\mathcal{C}$ must satisfy:
  - $\mathcal{C} \cap \mathcal{S} = \{\emptyset\}$.
  - If $C \in \mathcal{C}$ and $S \in \mathcal{S}$ then $C \cap S \in \mathcal{C}$.

Remark: (1) The elements of $\mathcal{S}$ are called **sense categories** and the elements of $\mathcal{C}$ are called **choice categories**.

(2) If $\mathcal{C}'$ is the ideal generated by $\mathcal{C}$, then $(U, \mathcal{S}, \mathcal{C}')$ is again a Mammen model, provided $(U, \mathcal{S}, \mathcal{C})$ is.

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  - If $C \in \mathcal{C}$ and $S \in \mathcal{S}$ then $C \cap S \in \mathcal{C}$.

Remark: (1) The elements of $\mathcal{S}$ are called **sense categories** and the elements of $\mathcal{C}$ are called **choice categories**.

(2) If $\mathcal{C}'$ is the ideal generated by $\mathcal{C}$, then $(U, \mathcal{S}, \mathcal{C}')$ is again a Mammen model, provided $(U, \mathcal{S}, \mathcal{C})$ is.

Complete Mammen models

Given a Mammen structure $(U, \mathcal{S}, \mathcal{C})$, it is natural to wonder if we may need more "categories" (i.e., collections of subsets of U) to describe every possible category in the universe (i.e., subset of U).

The notion of a **complete** Mammen structure seeks to add the requirement that every "category" is **finitely described**[1] given $\mathcal{S}$ and $\mathcal{C}$:

Definition

A Mammen model $(U, \mathcal{S}, \mathcal{C})$ is called **complete** if *every* $X \subseteq U$ can be written as

$$X = S \cup C$$

for some $S \in \mathcal{S}$ and $C \in \mathcal{C}$.

Remark: In a complete Mammen model, $\mathcal{C}$ is automatically an ideal.

[1] In Mammen's terminology, "finitely described" is called "decidable".

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Do complete models exist?

Question: Do complete Mammen models exist?

In the early 1990s, Jørgen Hoffmann-Jørgensen, who was a professor of mathematics in Aarhus, explored this question together with Jens Mammen. Using the Axiom of Choice, he was able to answer it:

Theorem (Hoffmann-Jørgensen, 2000. Uses the Axiom of Choice)

There are complete Mammen models.

In fact, if $\mathcal{T}$ is **any** *perfect Hausdorff topology on an infinite set U, then there is a finer perfect Hausdorff topology $\mathcal{S} \supseteq \mathcal{T}$ such that*

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Hoffmann-Jørgensen's proof uses the idea of *maximal perfect topologies*[2]:

Definition

A perfect topology $\mathcal{T}$ (on some set U) is called a **maximal perfect topology** if **no** topology which is strictly finer than $\mathcal{T}$ is perfect.

A routine application of Zorn's lemma shows:

Any perfect topology is contained in a maximal perfect topology.

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Sketch of the proof of Hoffmann-Jørgensen's theorem

Hoffmann-Jørgensen's theorem follows from the " $\Longrightarrow$ " direction of:

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Proof of " $\Longrightarrow$ ": Given $X \subseteq U$, let

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Then:

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Hoffmann-Jørgensen's theorem follows from the " $\Longrightarrow$ " direction of:

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Important corollary

Corollary

There is a complete Mammen model with a **countable** *universe U.*

Proof: Take $U = \mathbb{Q}$, and let $\mathcal{S}$ be a maximal perfect topology on $\mathbb{Q}$ which is finer than the topology generated by open rational intervals. $\square$

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We now know that complete Mammen models exist, but a the price of having invoked the Axiom of Choice (AC).

- While I personally feel pretty relaxed about AC, it is a bit startling that we needed it to prove that complete Mammen models exist.
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Theorem A (T.-Mammen, 2021)

In **Cohen's first model**, *a model of set theory in which all axioms of ZF are true, but AC is false, any perfect topology can be extended to a maximal perfect topology.*

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As a counterpoint to this, we also show that **some** substantial fragment of AC **is** needed to obtain a complete Mammen model:

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> - **The Boolean prime ideal theorem:** *Any ideal in a Boolean algebra can be extended to a prime ideal.*
> - **The Ultrafilter lemma:** *Any filter can be extended to an ultrafilter.*
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Joyfully, Theorem B is not heavy set theory. Instead, we need a basic measure-theoretic ingredient and a simple combinatorial ingredient (next two slides).

But first we switch our perspective a bit:

- From our point of view, rather than working with $\mathbb{R}$, it is much easier to work with **Cantor space**,

$$2^{\mathbb{N}} = \{f : \mathbb{N} \to \{0,1\} : f \text{ is a function}\}.$$

- We equip Cantor space with the measure μ, which is the product of $\mathbb{N}$ copies of the $(\frac{1}{2}, \frac{1}{2})$-measure on the two-point space $\{0,1\}$.
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Measure-theoretic ingredient: E_0-invariance

The following Lemma is well-known:

Lemma

Let $A \subseteq 2^{\mathbb{N}}$. Suppose that A is closed under finite changes, that is:

if $x, y \in A$ differ only finitely, and $x \in A$, then $y \in A$.

Suppose further that A is μ-measurable.

Then either $\mu(A) = 0$ or $\mu(A) = 1$.

Remark: Very often in the literature, invariance under finite changes is called "E_0-invariance".

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Combinatorial ingredient

Proposition (Proposition to remember!)

Let $(U, \mathcal{S}, \mathcal{C})$ be a Mammen model. Suppose there is $X \subseteq U$ such that:

$(*)$ *For any $V \in \mathcal{S} \setminus \{\emptyset\}$ the sets $V \cap X$ and $V \setminus X$ are infinite.*

Then $(U, \mathcal{S}, \mathcal{C})$ is not complete.

Proof:

- If $(U, \mathcal{S}, \mathcal{C})$ were complete, then

$$X = S \cup C$$

  for some $S \in \mathcal{S}$ and $C \in \mathcal{C}$.
- By $(*)$ we can't have $S \neq \emptyset$ (since $S \subseteq X$), so we must have $X = C$.
- A similar argument shows $U \setminus X \in \mathcal{C}$.
- Now $U = X \cup (U \setminus X) \in \mathcal{C}$, contradicting that $\mathcal{S} \cap \mathcal{C} = \{\emptyset\}$.

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Proposition (Proposition to remember!)

Let $(U, \mathcal{S}, \mathcal{C})$ be a Mammen model. Suppose there is $X \subseteq U$ such that:

$(*)$ *For any $V \in \mathcal{S} \setminus \{\emptyset\}$ the sets $V \cap X$ and $V \setminus X$ are infinite.*

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We are now ready to prove Theorem B. Recall what it says again:

Theorem B (T.-Mammen, 2021)

Assume ZF+"all subsets of $\mathbb{R}$ are Lebesgue measurable" (alternatively, Baire measurable). Then there is **no** *complete Mammen model with a* **countable** *universe.*

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Proof of Theorem B: *Assume all subsets of $2^{\mathbb{N}}$ are μ-measurable.*

To make life easier, we will identify $2^{\mathbb{N}}$ with $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$).

For $A \subseteq \mathbb{N}$, let $A^c = \mathbb{N} \setminus A$.

We let $\rho : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be the complementation function

$$\rho(A) = A^c.$$

Easy fact: ρ is μ-preserving, i.e. if $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ then $\mu(\rho(\mathcal{A})) = \mu(\mathcal{A})$.

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Proof [3]: Suppose not. Let $A \notin \mathcal{A}_n \cup \rho(\mathcal{A}_n)$.

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[3] The proof only uses that $\mathcal{S}$ is a perfect topology!

Sketch of Theorem B (cont'd)

Claim 1: With

$$\mathcal{A}_n = \{A \subseteq \mathbb{N} : (\forall V \in \mathcal{S})\ n \in V \implies |V \cap A| = \aleph_0\}$$

as on the previous slide, we have

$$\mathcal{P}(\mathbb{N}) = \mathcal{A}_n \cup \rho(\mathcal{A}_n).$$

Proof [3]: Suppose not. Let $A \notin \mathcal{A}_n \cup \rho(\mathcal{A}_n)$.

- By definition of $\mathcal{A}_n$ we can find $V, V' \in \mathcal{S}$ such that $n \in V$ and $n \in V'$ such that $A \cap V$ and $A^c \cap V'$ are finite.
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Claim 2: $\mu(\mathcal{A}_n) = 1$ and $\mu(\rho(\mathcal{A}_n)) = 1$.

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Finally, we can finish the proof of Theorem B:

- By Claim 2, we have that

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- If $V \in \mathcal{S}$ is non-empty, then there is some $n \in V$, and so by definition of $\mathcal{A}_n$ we must have that

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- So for any $V \in \mathcal{S} \setminus \{\emptyset\}$, the sets $V \cap X$ and $V \setminus X$ are infinite.
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Corollary in Solovay's model of ZF

In Solovay's model, where all sets are Lebesgue measurable, Theorem B gives the following corollary as a counterpoint to Theorem A:

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Cardinal invariants and questions

We will now make some final comments about the **cardinality** of $\mathcal{S}$ (the topology, i.e., the family of sense categories) in complete Mammen models (with universe $\mathbb{N}$).

Mammen has proved the following (using the "Proposition to remember"):

Theorem (Mammen, 1980s)

If $(U, \mathcal{S}, \mathcal{C})$ is a complete Mammen model, then $|\mathcal{S}| > \aleph_0$.

Since if $U = \mathbb{N}$ we have $\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})$, we get:

Corollary

The Continuum Hypothesis, CH, i.e., $2^{\aleph_0} = \aleph_1$, implies that the cardinality of $\mathcal{S}$ in a complete Mammen model with universe $\mathbb{N}$ is $\aleph_1$.

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Martin's axiom result

Mammen asked if $|\mathcal{S}| = 2^{\aleph_0}$ **always** holds, *even if CH is false* (i.e., even if $2^{\aleph_0} > \aleph_1$), for any $\mathcal{S}$ in a complete Mammen model with universe $\mathbb{N}$.

It turns out that this is **not** so: The answer depends heavily on which model of set theory we consider.

A positive answer can be given, assuming Martin's Axiom:

Theorem (T.-Mammen, 2021)

Assume **Martin's Axiom** *holds (along with all the usual axioms of ZFC). Then $|\mathcal{S}| = 2^{\aleph_0}$ for any $\mathcal{S}$ in a complete Mammen model with universe $\mathbb{N}$.*

In fact, a better result could be stated: $|\mathcal{S}| \geq \mathrm{add}(\mathrm{BP})$ (the additivity of the ideal of meager sets).

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Result in the Baumgartner-Laver model

However, a **negative** answer to Mammen's question can be provided in the so called "Baumgartner-Laver model".

The Baumgartner-Laver model is a model of ZFC in which $2^{\aleph_0} = \aleph_2$, and so the Continuum Hypothesis fails in this model. The model is obtained using iterated forcing by adding $\aleph_2$ Sacks reals, starting with a model of CH.

Theorem (T.-Mammen, 2021)

In the Baumgartner-Laver model of ZFC, in which $2^{\aleph_0} = \aleph_2$ holds, there is a complete Mammen model $(\mathbb{N}, \mathcal{S}, \mathcal{C})$ with

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Open questions

There are a number of questions still left open. The most interesting are:

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1. *Does "there exists a non-principal ultrafilter" imply that there exists a complete Mammen model?*
2. *Does the ultrafilter lemma ("all filters can be extended to an ultrafilter") imply the existence of a complete Mammen model?*
3. *Equivalent to the previous: Does the compactness theorem in first order logic imply the existence of a complete Mammen model?*
4. *Does the existence of a complete Mammen model imply that there is a maximal perfect topology?*

Finally, the last two theorems, about the cardinality of $\mathcal{S}$ in complete Mammen models, hints at a general project of determining the spectrum of possible cardinalites of such $\mathcal{S}$ (at least with the underlying set being $\mathbb{N}$).

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