Borel fractional graph theory (joint w/ Felix Weilacher)

Descriptive combinatorics:

graph theory with Borel, measurable,

Baire-measurable, etc. constructions.

- G a Borel graph on a Polish space, maximum degree $d \in \mathbb{N}$

  ↳ $\leq d$.

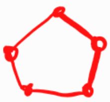
Borel chromatic # of G $= \chi_B(G) :=$ min $q \in \mathbb{N}$ s.t. there exist

Borel independent sets $I_1, \ldots, I_q \subseteq V(G)$ with $\bigcup_{i=1}^{q} I_i = V(G)$.

↳ $\in I_i$ $\in I_i$

Kechris-Solecki-Todorcevic '99: $\chi_B(G) \leq d+1$.

Marks '16: $\exists$ acyclic G, max. degree d s.t. $\chi_B = d+1$.

↳ no 

$d+1$

Brooks '47: $d \geq 3$, G has no $(d+1)$-clique $\Rightarrow \chi \leq d$.

Cf: Conley-Marks-Tucker-Drob '16:

[$d \geq 3$, G has no $(d+1)$-clique $\Rightarrow \chi_B \leq d$

modulo a null / meager set of vertices.]

fixed prob. measure μ on $V(G)$.

Extra "geometric" assumptions that upgrade

measurable / Baire-measurable combi. to Borel.

E.g.: G is component-finite $\Rightarrow$  "Borel combi

= ordinary combi"

boring!

Hyperfinite graphs: G is hyperfinite if there are Borel comp.-finite graphs $G_0 \subseteq G_1 \subseteq \dots$ s.t. $\bigcup_{i=0}^{\infty} G_i = G$

Not useful for Borel combi (Conley-Jackson-Marks-Seward-Tucker-Drob '20)

E.g.: $\exists$ hyperfinite acyclic G with $\chi_B = d+1$.

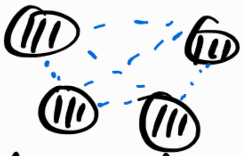
Other notions:

- G has subexponential growth (Conley-Tamuz, Csóka-Grabowski-Máthé-Pikhurko-Tyros, Thornton, AB-Dhawan, ...)

- G has finite asymptotic separation index (asi)

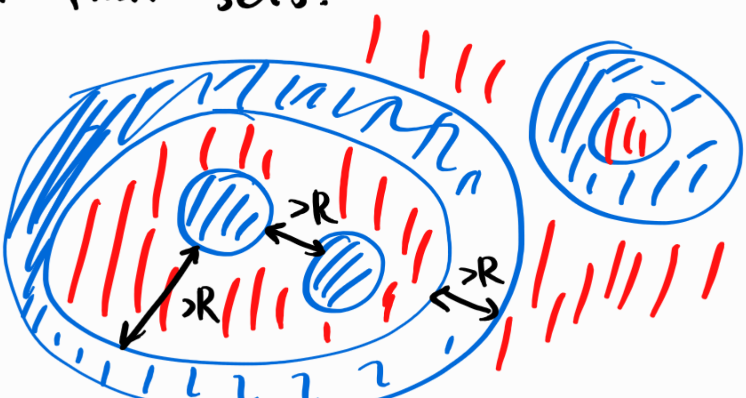
(Conley-Jackson-Marks-Seward-Tucker-Drob '20):

$U \subseteq V(G)$ is component-finite if $G[U]$ is a component-finite graph.



$si(G) \leq s$ if $V(G)$ can be partitioned into $s+1$ Borel component-finite sets.

asi ≤ 1:



$asi(G) \leq s$ if for

all $R > 0$, $si(G^R) \leq s$

all $R \in \mathbb{N}$, $si(G^{R}) \le 3$

connect every pair of vertices at distance $\le R$

* Mod meager sets: $asi \le 1$ for all locally finite Borel graphs
* Mod null: $asi \le 1 \iff asi < \infty \iff$ hyperfinite
* G is a Schreier graph of a Borel action of a finitely gen. group Γ where Γ is polycyclic, the lamplighter group, solvable + linear$/\mathbb{Q}$, etc...

  exp. growth

  $\Rightarrow asi(G) \le 1$.

OPEN: $asi < \infty \Rightarrow asi \le 1$?

OPEN: subexp. growth $\Rightarrow asi < \infty$?

(polynomial growth $\Rightarrow asi \le 1$, AB-Yu)

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max. degree $d \ge 3$, no $(d+1)$-clique

$$\left.\begin{array}{l}\text{subexp. growth} \xRightarrow{\text{CMT-D'16}} \\ asi < \infty \xRightarrow{\text{AB-Weilacher}}\end{array}\right\} \chi_B(G) \le d.$$

Pancake Property

* Related to toast (Conley-Miller '16, ----)
* Elek-Timár "strong almost finiteness"

Defn: An ε-pancake in G: A sequence $U_1, \dots, U_n \subseteq V(G)$ of Borel component-finite sets s.t. every vtx $v \in V(G)$ belongs to $\ge (1-\varepsilon)n$ of these sets.

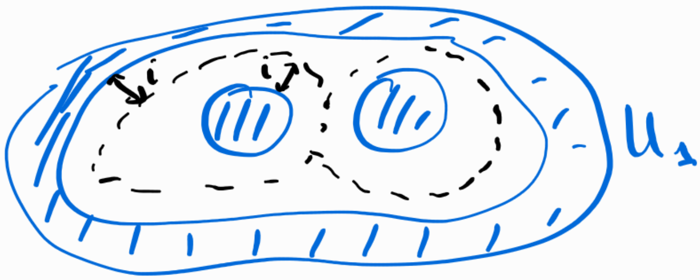
U_1

G has Pancake Property (P)

if $\forall \varepsilon > 0$, it has an ε-pancake.

Thm (AB-Weilacher) asi $< \infty \Rightarrow$ (P)

Picture for asi $= 1$:



Removing vertices at fixed distance i from U_1 leaves a comp.-finite set.

Also, subexp. growth $\Rightarrow$ (P).

("ball carving" from CS, Linial-Saks '93, Bartal '96, Balliu-Kuhn-Olivitti '21, ...)

G Schreier graph of a Borel action of an amenable group $\Rightarrow$ (P)

Rmks: Mod null: (P) $\Leftrightarrow$ hyperfinite

Mod meager: (P) holds.

Fractional graph theory (Meehan '18)

fractional colorings

fractional matchings

Borel k-fold q-coloring: Borel indep. sets $I_1, \ldots, I_q \subseteq V(G)$ s.t. every vtx is covered by $\geq k$ of them.

$\chi_{B,k}(G) :=$ min q s.t. $\exists$ k-fold q-coloring.

$$\chi^*_B(G) := \inf_{k \geq 1} \frac{\chi_{B,k}(G)}{k} \leq \chi_B(G).$$

Borel fractional chrom. #

Example:

$\chi = 3$ $\quad \chi^* = \frac{5}{2} < 3.$

2-fold 5-coloring

max deg $d \geq 3$, no $(d+1)$-clique $\Rightarrow \chi^*_B(G) \leq d.$

$(\underbrace{\text{Bousquet-Esperet-Pirot}}_{\text{algorithm}} + \text{AB})$

[**Thm** (AB-Weilacher) (P) $\Rightarrow \chi^*_B(G) = \chi^*(G)$.]

Corollary: Modulo meager sets, $\chi^*_B = \chi^*$.

Pf sketch: G has a k-fold q-coloring

(P) $\Rightarrow$ ε-pancake $U_1, \ldots, U_n$.

comp.-finite

$I_{11}, I_{12}, \ldots, I_{1q}$ Borel k-fold q-coloring of $G[U_1]$.

$\cdots\cdots$

$I_{n1}, I_{n2}, \ldots, I_{nq}$ Borel k-fold q-coloring of $G[U_n]$.

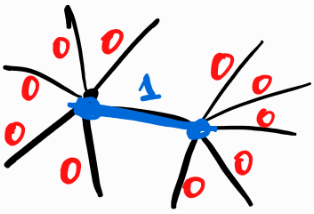
$(1-\varepsilon)nk$-fold nq-coloring $\rightsquigarrow \chi^*_B(G) \leq \frac{nq}{(1-\varepsilon)kn} = \frac{q}{(1-\varepsilon)k}$

$\overset{\varepsilon \to 0}{\Longrightarrow} \chi^*(G) \leq \frac{q}{\ }$

$\chi_B(G) = k$.

A fractional matching: $f: E(G) \to [0,1]$ s.t. the sum around every vtx is ≤ 1.

A matching: a $\{0,1\}$-valued fractional matching.



Perfect: the sum around every vtx is $= 1$.

[Thm (AB-W) Assume (P). G has a fractional PM $\Rightarrow$ $\forall \varepsilon > 0$ Borel fractional matching s.t. the sum around every vtx is $\geq 1 - \varepsilon$.]

Thm (AB-W): $\exists$ acyclic Borel graph G of max deg. 3 with a PM but no Borel fractional PM off a meager set. Moreover, for any $f: \mathbb{N} \to \mathbb{N}$

(P), asi ≤ 1

s.t. $f(0) = 1$, $f(1) \geq 3$, $\frac{f(n)}{n} \to \infty$, can make G have growth rate $\leq f$.

But: If G is of linear growth, has a fractional PM, then G has a Borel frac. PM off a meager set.

OPEN: Is this necessary?