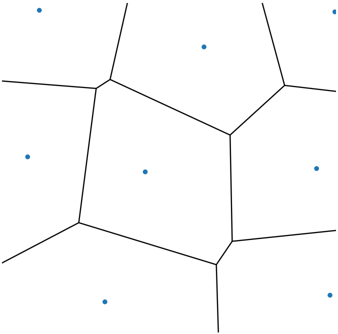
Hyperfiniteness of Borel graphs of slow intermediate growth

Andrew Marks, work in progress with Jan Grebík, Václav Rozhoň, and Forte Shinko

Caltech Logic Seminar, Oct 2 2024



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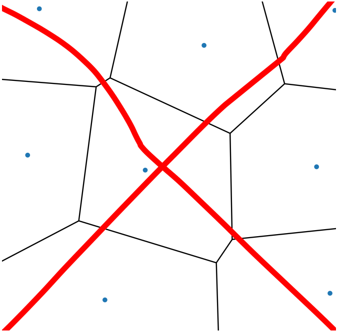

Voronoi regions are not a great way to subdivide space.

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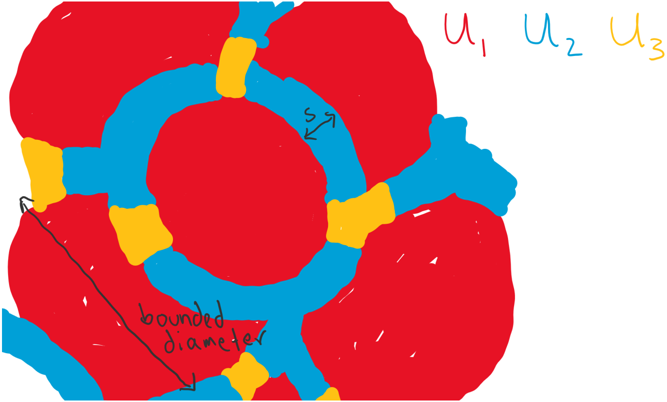
$A, B \subseteq X$ are s-**disjoint** if $\rho(x, y) > s$ for every $x \in A$ and $y \in B$. The **diameter** of $A \subseteq X$ is $\operatorname{diam}(A) = \sup\{\rho(x, y) \colon x, y \in A\}$.

Ball carving with parameter ϵ

Suppose (X, ρ) has growth bounded by a subexponential function h so $|B_r(x)| \leq h(r)$ for all x, r.

We will make families $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of X, where elements of $\mathcal{U}_i$ are pairwise s-disjoint and have bounded diameter.



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- Add $B_{r-s}(x)$ to $\mathcal{U}_1$, and remove $B_r(x)$ from our space X. Since $B_r(x) \leq (1+\epsilon)B_{r-s}(x)$, we have $(B_r(x) \setminus B_{r-s}(x)) \leq \epsilon B_{r-s}(x)$. Each $y \in B_r(x) \setminus B_{r-s}(x)$ we haven't covered "blames" $1/\epsilon$ many points of $B_{r-s}(x)$.

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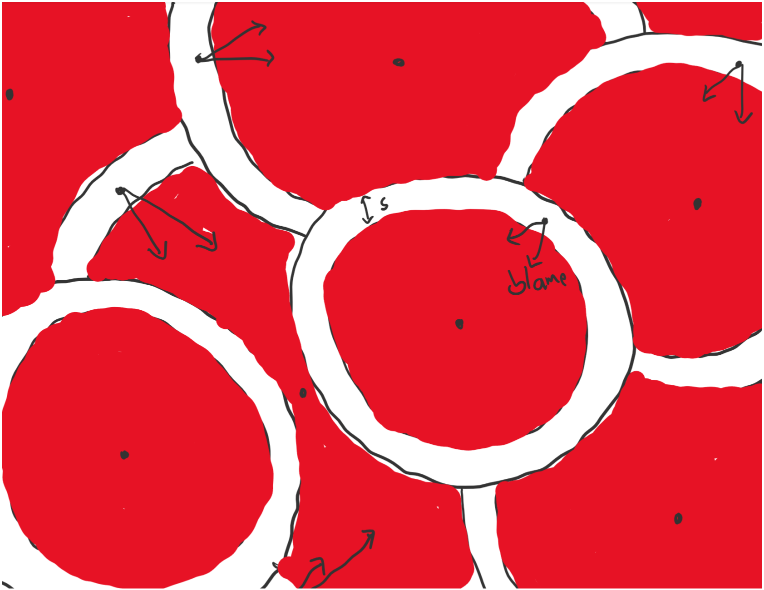
- Repeat this process starting at some remaining point y to add another ball to $\mathcal{U}_1$. Iterate until nothing is left. The elements of $\mathcal{U}_1$ all have diameter $\leq 2t$ and are pairwise s-disjoint.

Picture of $\mathcal{U}_1$



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- Recursively make $\mathcal{U}_{i+1}$ by working in the subspace $X \setminus \bigcup(\mathcal{U}_1 \cup \ldots \cup \mathcal{U}_i)$, and doing exactly the same process. Because we still have the growth upper bound of h on this subspace, all the calculations are exactly the same so the diameters of the sets in U_{i+1} are still at most $2t$.

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- If x is not added to $U_1, \ldots \mathcal{U}_n$ at step n, then the blame tree starting at x has height $n+1$, and each $y \in \bigcup \mathcal{U}_i$ in the tree has blamed $1/\epsilon$ points of $\bigcup \mathcal{U}_{i-1}$ of distance $\leq 2t$. So the tree has at least $(1/\epsilon)^n$ leaves which are all contained in $B_{2nt}(x)$.

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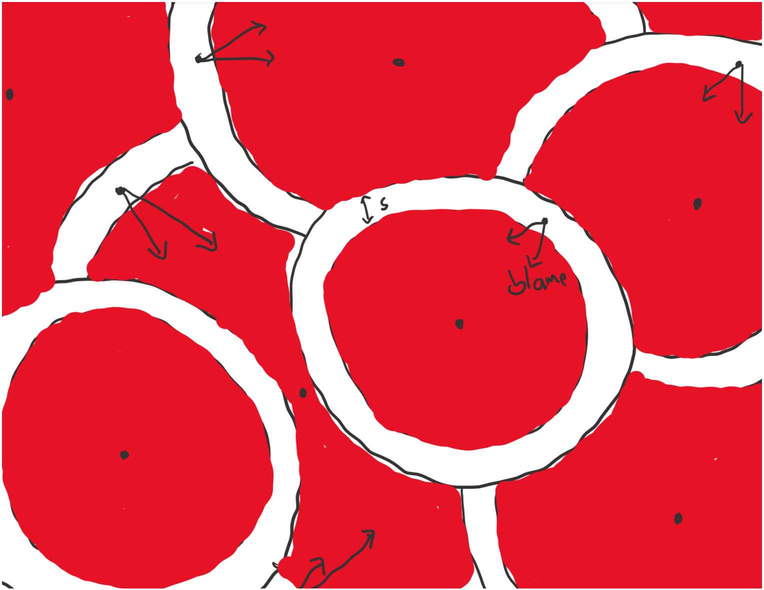
- Since h is subexponential, there is a finite m so $(1/\epsilon)^m > h(2mt)$ so the process must cover every element of the space after n steps where $n < m$. $\mathcal{U}_1, \ldots, \mathcal{U}_n$ is a cover of the space by s-disjoint sets.

Picture of the construction of $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$



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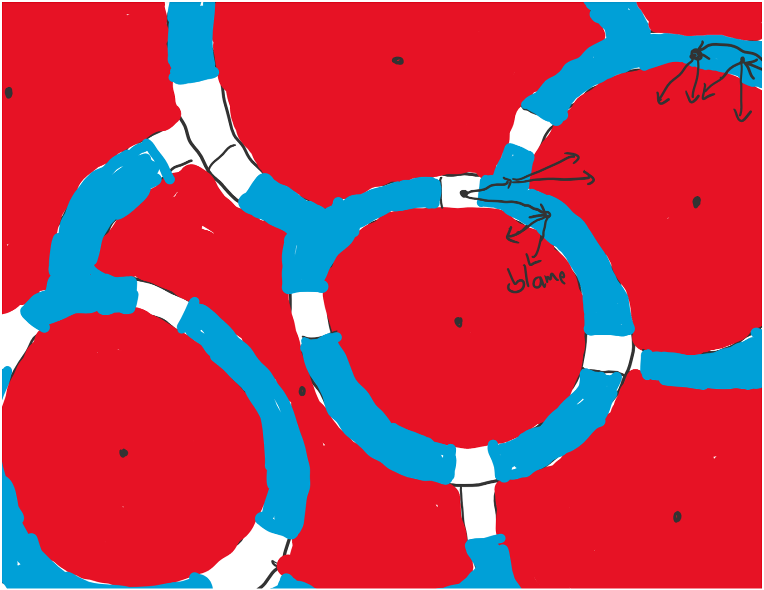

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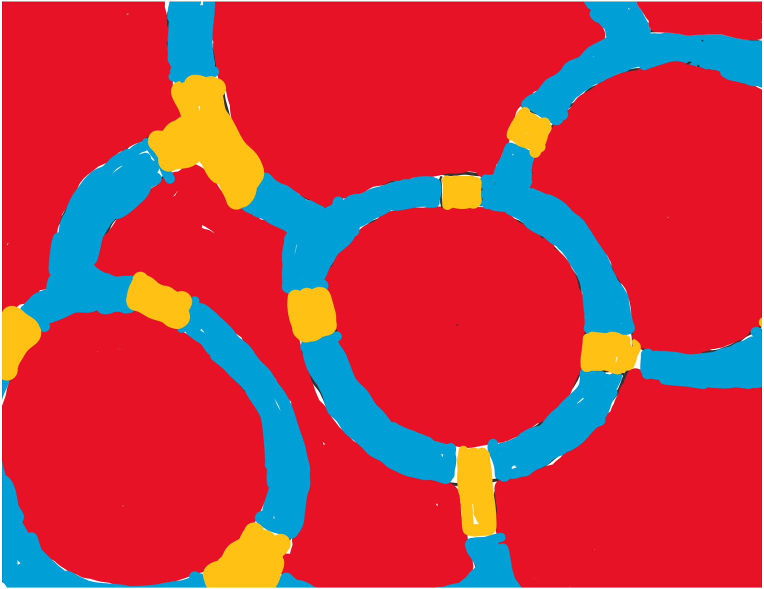

Bounds for polynomial growth

Suppose $B_r(x) \leq r^d$. Then for each s, let $\epsilon = 1/s^2$ and $t = s^4$. Then $(1 + 1/\epsilon)^{t/s} \approx e^s > s^{4d}$ provided s is sufficiently large. So the sets in the covers have diameter at most $2t = 2s^4$. The size of the blame tree after n steps is $(1/\epsilon)^n = s^{2n}$. But it is contained in a ball of radius $2ns^4$ which contains at most $(2ns^4)^d$ points. So there is a constant n so for all s, $s^{2n} > (2ns^4)^d = O(s^{4d})$. So for all s, there is a constant n so that there is a cover $\mathcal{U}_1, \ldots, \mathcal{U}_n$ by s-disjoint families of sets of diameter at most $2s^4$.

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Corollary (Bernshteyn-Yu)

If G is a Borel graph of polynomial growth, then the graph metric ρ_G has finite Borel asymptotic dimension. Hence G is hyperfinite.

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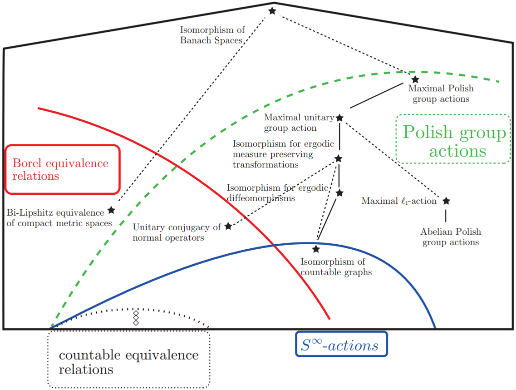

Open questions about hyperfiniteness

There are many open questions about hyperfiniteness:

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  - Some of the above proofs use intricate understandings of Følner set in these groups and their tilings, but we'd like easier geometric/graph theoretic tools.

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"Large-scale" analog of covering dimension. $\mathrm{asdim}(X) \leq d$ if

- For every (large) $r > 0$, there is a cover $\mathcal{U}$ of X by sets of uniformly bounded diameter, so every r-ball $B_r(x)$ intersects at most $d + 1$ elements of $\mathcal{U}$.

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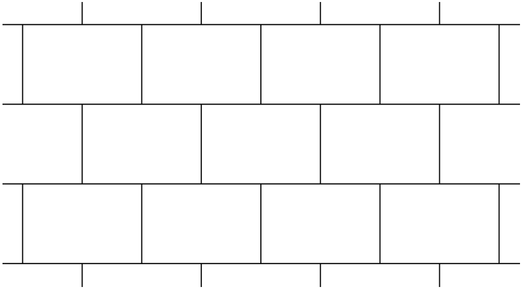

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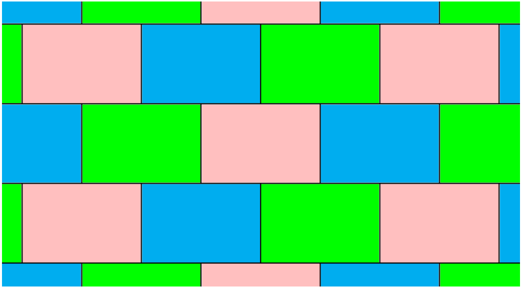

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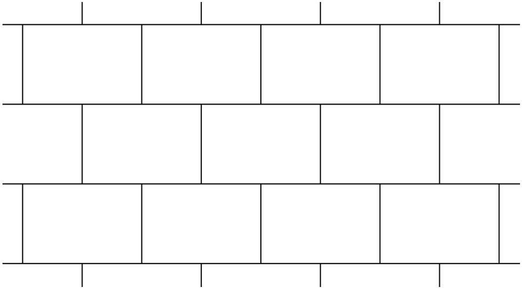

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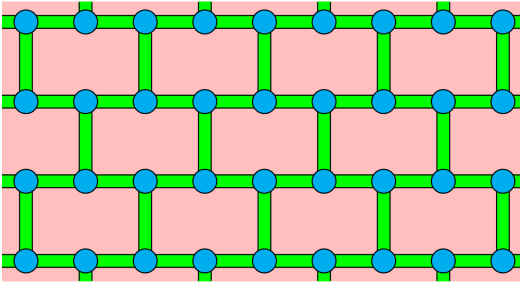

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- $\text{asdim}(\mathbb{F}_n) \leq 1$.
- If $X \subseteq Y$, then $\text{asdim}(X) \leq \text{asdim}(Y)$. So groups that contain (coarse embeddings of) $\mathbb{Z}^n$ for every n have infinite asymptotic dimension. E.g. $\mathbb{Z} \wr \mathbb{Z}$, or Thompson's group F, or the Grigorchuk group G.

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Theorem (Conley, Jackson, Seward, M., Tucker-Drob)

Suppose (X, ρ) is a Borel proper extended metric space with finite asymptotic dimension. Then E_ρ is hyperfinite, where $x \, E_\rho \, y$ if $\rho(x, y) < \infty$.

Generalizing finite asymptotic dimension

Suppose (X,ρ) is a proper extended metric space. The **dimension function** of (X,ρ) is the function $c\colon (0,\infty)\times(0,\infty)\to\mathbb{N}$ where $c(r,s)$ is the least n so that there is a cover $\mathcal{U}_1,\ldots,\mathcal{U}_n$ of X by n families of s-disjoint sets of diameter at most r.

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In this language, (X, ρ) has asymptotic dimension d if for every s there exists an r so that $c(s, r) \leq d + 1$.

Slow dimension growth implies hyperfiniteness

A slow enough growing dimension function still implies hyperfiniteness even if $\sup_s \inf_r c(s,r) = \infty$:

Theorem

Suppose (ρ, X) is Borel proper extended metric space with Borel dimension function c. Suppose there exists sequences $(a_n)_{n\in\omega}$, $(r_n)_{n\in\omega}$, and $(s_n)_{n\in\omega}$ of positive integers such that $s_n \geq 4a_{n+1}r_{n-1}$, and $a_n \geq c(r_n, s_n)$. Then (X, ρ) is hyperfinite.

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Proof sketch: Given a collection $\mathcal{U}$ of subsets of X, let $E_{\mathcal{U}}$ be the equivalence relation where $x \mathrel{E_{\mathcal{U}}} y$ if for all $U \in \mathcal{U}$, $x \in U \leftrightarrow y \in U$. So if $\mathcal{U}$ is a cover of X by finite sets, then $E_{\mathcal{U}}$ has finite classes.

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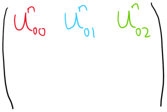
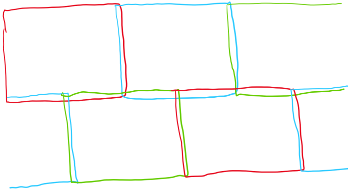
Say collections $\mathcal{U}, \mathcal{V}$ of subsets of X are have disjoint boundaries if for all $A \in \mathcal{U}$ and $B \in \mathcal{V}$, $B_1(A) \setminus A$ and $B_1(B) \setminus B$ are disjoint. Hence if $\rho(x,y) = 1$, then $x \, E_{\mathcal{U}} \, y$ or $x \, E_{\mathcal{V}} \, y$.

Slow dimension growth implies hyperfiniteness

Inductively build a sequence of $(a_{n+1} + 1) \times a_n$ matrices $(\mathcal{U}^n_{ij})$ where:

- The first row $\mathcal{U}^n_{10}, \ldots, U^n_{1a_n}$ is a Borel cover by disjoint sets of diameter at most r_n that are s_n-separated.





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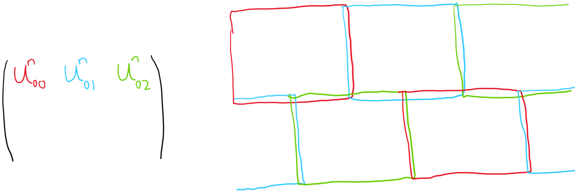

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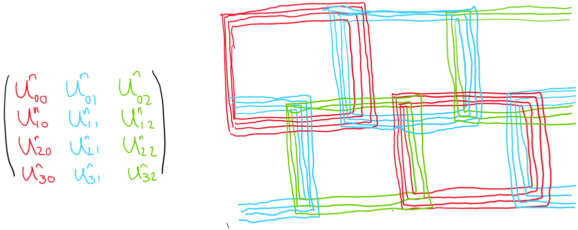

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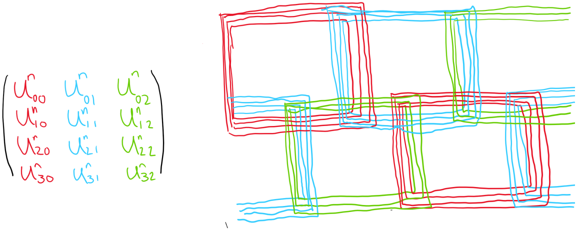

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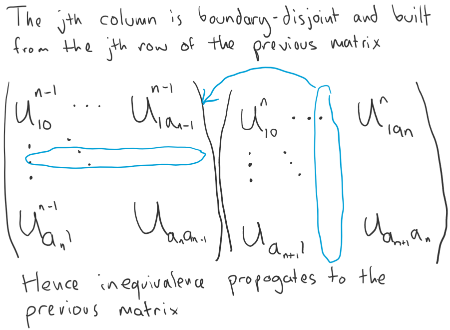
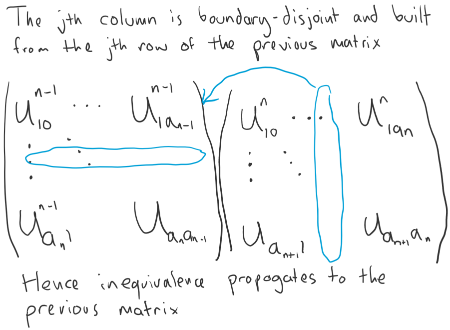
- The jth column in $(\mathcal{U}^n_{ij})$ has pairwise disjoint boundaries and each $U \in \mathcal{U}^n_{ij}$ is a union of sets from the jth row of the previous matrix $\mathcal{U}^{n-1}_{ij}$.



Suppose $\rho(x, y) = 1$. Because each column has disjoint boundaries, x and y are inequivalent in at most one equivalence relation $E_{\mathcal{U}_{ij}^n}$ from each column. Since the jth column of U_{ij}^{n+1} is built from the jth row of U_{ij}^n, inequivalence propagates backwards from the jth column to the jth row of previous matrix.



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Hence there can be at most a_0 matrices where x, y are inequivalent in the last row. Let $\mathcal{U}^n_{d+1} = \bigcup_i \mathcal{U}^n_{a_{n+1}i}$ be the cover from the last row, so x, y are $E_{\mathcal{U}^n_{d+1}}$-inequivalent for at most a_0 values of n. Letting $\mathcal{V}^n = \bigcup_{m \geq n} \mathcal{U}^m_{d+1}$, $E_{\mathcal{V}^n}$ witnesses that E_ρ is hyperfinite. $\square$

Finishing the proof

Theorem

Suppose (X, ρ) is a proper Borel extended metric space and there is a constant C so that $B_r(x) \leq C \exp(r^{0.15229})$. Then E_ρ is hyperfinite.

Proof idea: use ball carving to show that if $B_r(x) \leq C \exp(r^\gamma)$, then the Borel dimension function of (X, ρ) satisfies $c(r, s) \leq r^{\gamma/(1-\gamma)}$ provided that $s \ll r^{1-\gamma}$.

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Let $r_n = 2^{b^n}$, $s_n = r_n^{1-\gamma}$ and $a_n = 2^{\gamma/(1-\gamma)b^n}$ where $b = (1-\gamma)^2/(2\gamma)$. These satisfy the hypotheses of our earlier theorem on slow dimension growth giving hyperfiniteness, provided $(1-\gamma)^3/(2\gamma) \geq \gamma/(1-\gamma)((1-\gamma)^2/(2\gamma))^2 + 1$. Solving to find the root of this cubic we get $\gamma \approx 0.15229$.

What groups does this apply to?

Our results show that if Γ is a finitely generated group of growth slower than $\exp(n^{0.15229})$, then free actions of Γ are hyperfinite. What groups are there of superpolynomial growth bounded by $\exp(n^{0.15229})$?

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Conjecture (The gap conjecture, Grigorchuck 1990)

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Maybe some:
Tom Hutchcroft: "there is no compelling reason to believe the gap conjecture is true."

Open questions

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- Does subexponential growth imply finite asymptotic separation index?
- What Borel dimension functions imply hyperfiniteness? (There is currently a large gap between the logarithmic growth we know gives hyperfiniteness, and the exponential growth we know exists in non-μ-hyperfinite pmp graphs).

Thanks!