# Structurable equivalence relations and $\mathcal{L}_{\omega_1\omega}$ interpretations

Ronnie Chen (University of Michigan)
joint with Rishi Banerjee (UIC)

October 9, 2024

# Locally countable Borel combinatorics

…is the study of Borel combinatorial structures (e.g., trees, group actions) on standard Borel spaces (e.g., $\mathbb{R}$, $2^{\mathbb{N}}$) which can be partitioned into countable pieces.

# Locally countable Borel combinatorics

...is the study of Borel combinatorial structures (e.g., trees, group actions) on standard Borel spaces (e.g., $\mathbb{R}$, $2^{\mathbb{N}}$) which can be partitioned into countable pieces.

Example A Borel action $\mathbb{F}_2 \circlearrowright X$ induces a Schreier graph on $X$ which is a Borel graph with countable components.

If the action is free, then the graph is a forest (each component is a tree).

# Locally countable Borel combinatorics

…is the study of Borel combinatorial structures (e.g., trees, group actions) on standard Borel spaces (e.g., $\mathbb{R}$, $2^{\mathbb{N}}$) which can be partitioned into countable pieces.

Example A Borel action $\mathbb{F}_2 \curvearrowright X$ induces a Schreier graph on $X$ which is a Borel graph with countable components.

If the action is free, then the graph is a forest (each component is a tree).

This talk is about the *global* aspects of "all" locally ctbl Borel combinatorial structures.

# Locally countable Borel combinatorics

…is the study of Borel combinatorial structures (e.g., trees, group actions) on standard Borel spaces (e.g., $\mathbb{R}$, $2^{\mathbb{N}}$) which can be partitioned into countable pieces.

Example A Borel action $\mathbb{F}_2 \curvearrowright X$ induces a Schreier graph on $X$ which is a Borel graph with countable components.

If the action is free, then the graph is a forest (each component is a tree).

This talk is about the *global* aspects of "all" locally ctbl Borel combinatorial structures.

Definition A **countable Borel equivalence relation (CBER)** $E \subseteq X^2$ is a Borel equivalence relation with countable equivalence classes (the "countable pieces").

Instead of "Borel structures with countable pieces", we look at "Borel families of countable structures" on the classes of a CBER.

# The Feldman–Moore theorem

*Every CBER* $E \subseteq X^2$ *is generated by a Borel action* $\Gamma \curvearrowright X$ *of a countable group* $\Gamma$.

# The Feldman–Moore theorem

*Every CBER* $E \subseteq X^2$ *is generated by a Borel action* $\mathbb{F}_\omega \circlearrowright X$.

# The Feldman–Moore theorem

*Every CBER* $E \subseteq X^2$ *is generated by a Borel action* $\mathbb{F}_\omega \circlearrowright X$.
*In other words, we may put a transitive* $\mathbb{F}_\omega$*-action on each* $E$*-class in a Borel way.*

# The Feldman–Moore theorem

*Every CBER* $E \subseteq X^2$ *is generated by a Borel action* $\mathbb{F}_\omega \circlearrowright X$*.*
*In other words, we may put a transitive* $\mathbb{F}_\omega$*-action on each* $E$*-class in a Borel way.*

Proof. By Lusin–Novikov uniformization, $E = \bigcup_{n \in \mathbb{N}} f_n$ for Borel $f_n : X \to X$.

# The Feldman–Moore theorem

*Every CBER $E \subseteq X^2$ is generated by a Borel action $\mathbb{F}_\omega \circlearrowright X$.*
*In other words, we may put a transitive $\mathbb{F}_\omega$-action on each $E$-class in a Borel way.*

Proof. By Lusin–Novikov uniformization, $E = \bigcup_{n \in \mathbb{N}} f_n$ for Borel $f_n : X \to X$.
In other words, $C^2 = \bigcup_n f_n$ for $f_n : C \to C$ for each $C \in X/E$ in a Borel way.

# The Feldman–Moore theorem

*Every CBER $E \subseteq X^2$ is generated by a Borel action $\mathbb{F}_\omega \curvearrowright X$.*
*In other words, we may put a transitive $\mathbb{F}_\omega$-action on each $E$-class in a Borel way.*

Proof. By Lusin–Novikov uniformization, $E = \bigcup_{n \in \mathbb{N}} f_n$ for Borel $f_n : X \to X$.
In other words, $C^2 = \bigcup_n f_n$ for $f_n : C \to C$ for each $C \in X/E$ in a Borel way.
Let $f_{ij} := f_i \cap f_j^{-1}$; then $C^2 = \bigcup_{i,j} f_{ij}$ where the $f_{ij}$ are partial bijections.

# The Feldman–Moore theorem

*Every CBER $E \subseteq X^2$ is generated by a Borel action $\mathbb{F}_\omega \curvearrowright X$.*
*In other words, we may put a transitive $\mathbb{F}_\omega$-action on each $E$-class in a Borel way.*

Proof. By Lusin–Novikov uniformization, $E = \bigcup_{n\in\mathbb{N}} f_n$ for Borel $f_n : X \to X$.
In other words, $C^2 = \bigcup_n f_n$ for $f_n : C \to C$ for each $C \in X/E$ in a Borel way.
Let $f_{ij} := f_i \cap f_j^{-1}$; then $C^2 = \bigcup_{i,j} f_{ij}$ where the $f_{ij}$ are partial bijections.
Since $X$ is standard Borel, there is a family of separating Borel sets $(U_k)_{k\in\mathbb{N}}$.

# The Feldman–Moore theorem

*Every CBER $E \subseteq X^2$ is generated by a Borel action $\mathbb{F}_\omega \curvearrowright X$.*
*In other words, we may put a transitive $\mathbb{F}_\omega$-action on each $E$-class in a Borel way.*

Proof. By Lusin–Novikov uniformization, $E = \bigcup_{n\in\mathbb{N}} f_n$ for Borel $f_n : X \to X$.
In other words, $C^2 = \bigcup_n f_n$ for $f_n : C \to C$ for each $C \in X/E$ in a Borel way.
Let $f_{ij} := f_i \cap f_j^{-1}$; then $C^2 = \bigcup_{i,j} f_{ij}$ where the $f_{ij}$ are partial bijections.
Since $X$ is standard Borel, there is a family of separating Borel sets $(U_k)_{k\in\mathbb{N}}$. Let

$$f_{ijk} := f_i \cap f_j^{-1} \cap ((U_k \times X) \triangle (X \times U_k)),$$

$$\text{i.e., } f_{ijk}(x) = y \ :\Longleftrightarrow\ (f_i(x) = y) \wedge (x = f_j(y)) \wedge (U_k(x) \leftrightarrow \neg U_k(y)).$$

Then $C^2 = \bigcup_{i,j,k} f_{ijk} \cup (=_C)$ for partial bijs $f_{ijk}$ w/ disjoint domains & images.

# The Feldman–Moore theorem

*Every CBER $E \subseteq X^2$ is generated by a Borel action $\mathbb{F}_\omega \curvearrowright X$.*
*In other words, we may put a transitive $\mathbb{F}_\omega$-action on each $E$-class in a Borel way.*

Proof. By Lusin–Novikov uniformization, $E = \bigcup_{n \in \mathbb{N}} f_n$ for Borel $f_n : X \to X$.
In other words, $C^2 = \bigcup_n f_n$ for $f_n : C \to C$ for each $C \in X/E$ in a Borel way.
Let $f_{ij} := f_i \cap f_j^{-1}$; then $C^2 = \bigcup_{i,j} f_{ij}$ where the $f_{ij}$ are partial bijections.
Since $X$ is standard Borel, there is a family of separating Borel sets $(U_k)_{k \in \mathbb{N}}$. Let

$$f_{ijk} := f_i \cap f_j^{-1} \cap ((U_k \times X) \triangle (X \times U_k)),$$

$$\text{i.e., } f_{ijk}(x) = y \;:\Longleftrightarrow\; (f_i(x) = y) \wedge (x = f_j(y)) \wedge (U_k(x) \leftrightarrow \neg U_k(y)).$$

Then $C^2 = \bigcup_{i,j,k} f_{ijk} \cup (=_C)$ for partial bijs $f_{ijk}$ w/ disjoint domains & images. So

$$g_{ijk} := f_{ijk} \cup f_{ijk}^{-1} \cup (=_{\neg(\mathrm{dom}(f_{ijk}) \cup \mathrm{im}(f_{ijk}))})$$

$$\text{i.e., } g_{ijk}(x) = y \;:\Longleftrightarrow\; (f_{ijk}(x) = y) \vee (x = f_{ijk}(y)) \vee (x = y \notin \mathrm{dom}(f_{ijk}) \cup \mathrm{im}(f_{ijk}))$$

are countably many involutions covering each $C^2$, defined in a Borel way.

# The Feldman–Moore theorem

*Every CBER $E \subseteq X^2$ is generated by a Borel action $\mathbb{Z}_2^{*\omega} \circlearrowright X$.*
*In other words, we may put a transitive $\mathbb{Z}_2^{*\omega}$-action on each $E$-class in a Borel way.*

Proof. By Lusin–Novikov uniformization, $E = \bigcup_{n \in \mathbb{N}} f_n$ for Borel $f_n : X \to X$.
In other words, $C^2 = \bigcup_n f_n$ for $f_n : C \to C$ for each $C \in X/E$ in a Borel way.
Let $f_{ij} := f_i \cap f_j^{-1}$; then $C^2 = \bigcup_{i,j} f_{ij}$ where the $f_{ij}$ are partial bijections.
Since $X$ is standard Borel, there is a family of separating Borel sets $(U_k)_{k \in \mathbb{N}}$. Let

$$f_{ijk} := f_i \cap f_j^{-1} \cap ((U_k \times X) \triangle (X \times U_k)),$$

$$\text{i.e., } f_{ijk}(x) = y \;:\Longleftrightarrow\; (f_i(x) = y) \wedge (x = f_j(y)) \wedge (U_k(x) \leftrightarrow \neg U_k(y)).$$

Then $C^2 = \bigcup_{i,j,k} f_{ijk} \cup (=_C)$ for partial bijs $f_{ijk}$ w/ disjoint domains & images. So

$$g_{ijk} := f_{ijk} \cup f_{ijk}^{-1} \cup (=_{\neg(\mathrm{dom}(f_{ijk}) \cup \mathrm{im}(f_{ijk}))})$$

$$\text{i.e., } g_{ijk}(x) = y \;:\Longleftrightarrow\; (f_{ijk}(x) = y) \vee (x = f_{ijk}(y)) \vee (x = y \notin \mathrm{dom}(f_{ijk}) \cup \mathrm{im}(f_{ijk}))$$

are countably many involutions covering each $C^2$, defined in a Borel way.
Close under compositions. $\square$

# Structurings

Let $\mathcal{L}$ be a countable first-order language.

Definition A **Borel $\mathcal{L}$-structuring** $\mathcal{M}$ of a CBER $E \subseteq X^2$ is a family of countable $\mathcal{L}$-structures $(\mathcal{M}_C)_{C \in X/E}$ on each $E$-class $C$ such that "$C \mapsto \mathcal{M}_C$ is Borel".

# Structurings

Let $\mathcal{L}$ be a countable first-order language.

Definition A **Borel $\mathcal{L}$-structuring** $\mathcal{M}$ of a CBER $E \subseteq X^2$ is a family of countable $\mathcal{L}$-structures $(\mathcal{M}_C)_{C \in X/E}$ on each $E$-class $C$ such that "$C \mapsto \mathcal{M}_C$ is Borel".

Example A locally countable Borel graph $G \subseteq X^2$ is an $\mathcal{L}_{\text{graph}}$-structuring for $\mathcal{L}_{\text{graph}} = \{G\}$ (where $G$ is a binary relation symbol) of any CBER $E \supseteq G$.

Example A Borel $\Gamma$-action generating $E$ is an $\mathcal{L}_\Gamma$-structuring of $E$ for $\mathcal{L}_\Gamma = \{a_\gamma\}_{\gamma \in \Gamma}$ (where $a_\gamma$ is a unary function symbol).

# Structurings

Let $\mathcal{L}$ be a countable first-order language.

Definition A **Borel $\mathcal{L}$-structuring** $\mathcal{M}$ of a CBER $E \subseteq X^2$ is a family of countable $\mathcal{L}$-structures $(\mathcal{M}_C)_{C \in X/E}$ on each $E$-class $C$ such that "$C \mapsto \mathcal{M}_C$ is Borel".

Example A locally countable Borel graph $G \subseteq X^2$ is an $\mathcal{L}_{\text{graph}}$-structuring for $\mathcal{L}_{\text{graph}} = \{G\}$ (where $G$ is a binary relation symbol) of any CBER $E \supseteq G$.

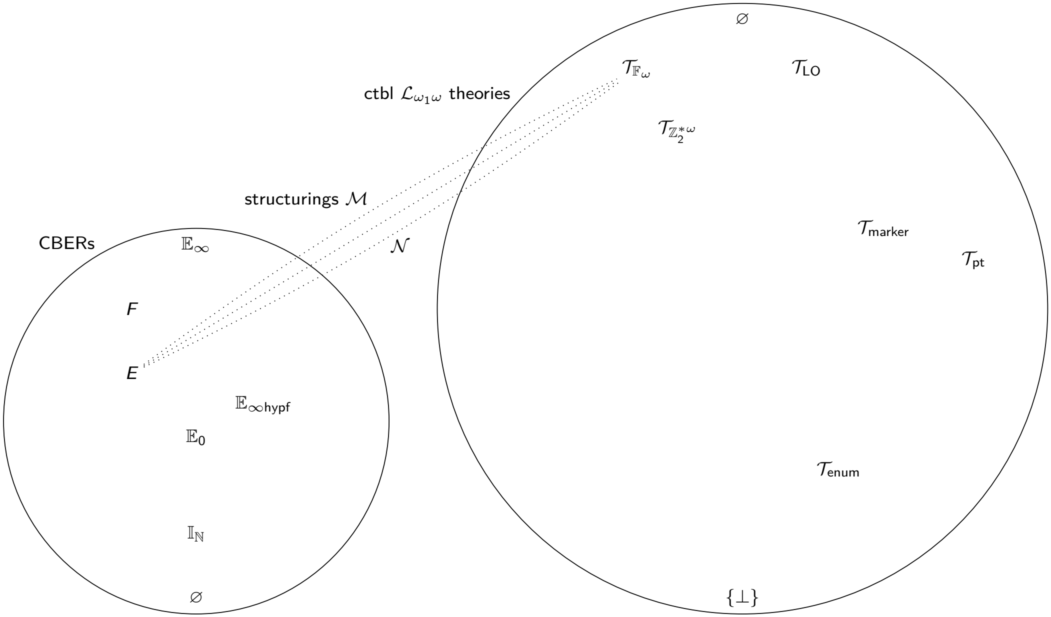
Example A Borel $\Gamma$-action generating $E$ is an $\mathcal{L}_\Gamma$-structuring of $E$ for $\mathcal{L}_\Gamma = \{a_\gamma\}_{\gamma \in \Gamma}$ (where $a_\gamma$ is a unary function symbol).

$\boldsymbol{\mathcal{L}_{\omega_1\omega}}$ is countably infinitary first-order logic with $\bigwedge_{n \in \mathbb{N}} \phi_n, \bigvee_{n \in \mathbb{N}} \phi_n$ (plus $\neg, \exists, \forall$).

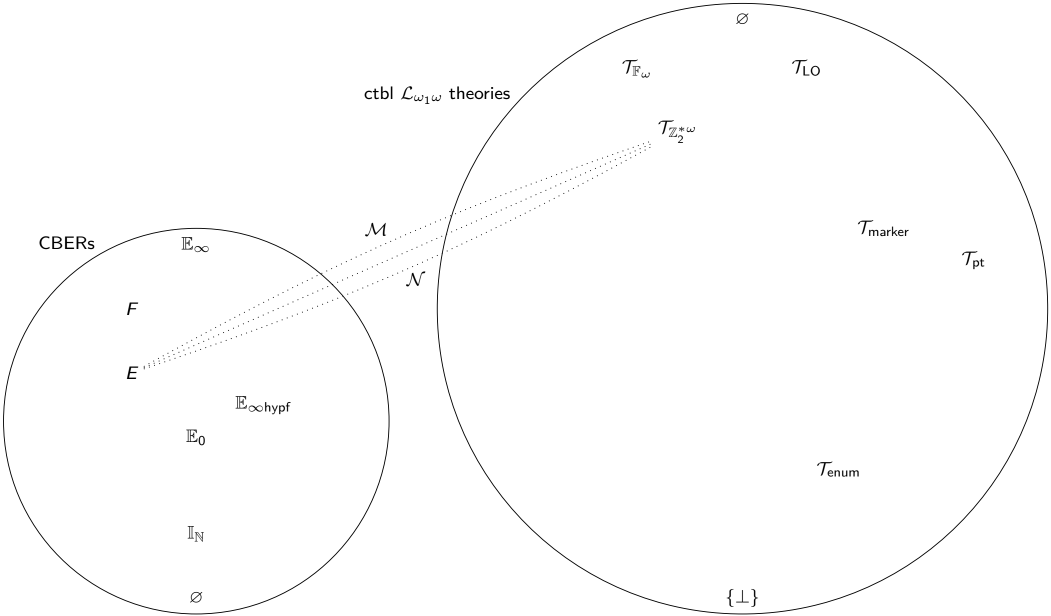
Example A Borel $\Gamma$-action generating $E$ is actually a structuring by models of

$$\mathcal{T}_\Gamma := \{\forall x\,(a_1(x) = x)\} \cup \{\forall x\,(a_\gamma(a_\delta(x)) = a_{\gamma\delta}(x)) \mid \gamma, \delta \in \Gamma\}$$
$$\cup \left\{\forall x, y \bigvee_{\gamma \in \Gamma} (a_\gamma(x) = y)\right\}.$$

For a (ctbl $\mathcal{L}_{\omega_1\omega}$) theory $\mathcal{T}$, a **$\mathcal{T}$-structuring** is an $\mathcal{L}$-structuring $\mathcal{M}$ s.t. each $\mathcal{M}_C \models \mathcal{T}$.



CBERs
$\mathbb{E}_\infty$
$F$
$E$
$\mathbb{E}_{\infty\mathsf{hypf}}$
$\mathbb{E}_0$
$\mathbb{I}_\mathbb{N}$
$\varnothing$
structurings $\mathcal{M}$
$\mathcal{N}$
ctbl $\mathcal{L}_{\omega_1\omega}$ theories
$\varnothing$
$\mathcal{T}_{\mathbb{F}_\omega}$
$\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{marker}}$
$\mathcal{T}_{\mathsf{pt}}$
$\mathcal{T}_{\mathsf{enum}}$
$\{\bot\}$


CBERs
$\mathbb{E}_\infty$
$F$
$E$
$\mathbb{E}_{\infty\mathsf{hypf}}$
$\mathbb{E}_0$
$\mathbb{I}_\mathbb{N}$
$\varnothing$
$\mathcal{M}$
$\mathcal{N}$
ctbl $\mathcal{L}_{\omega_1\omega}$ theories
$\varnothing$
$\mathcal{T}_{\mathbb{F}_\omega}$
$\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{marker}}$
$\mathcal{T}_{\mathsf{pt}}$
$\mathcal{T}_{\mathsf{enum}}$
$\{\bot\}$

# Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$ and $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in WLOG relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

# Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$ and $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in WLOG relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping $n$-ary $\mathcal{L}_1$-relns $R \longmapsto \mathcal{L}_2$-formulas $\alpha(R)(x_0, \ldots, x_{n-1})$

# Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$ and $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in WLOG relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping $n$-ary $\mathcal{L}_1$-relns $R \longmapsto \mathcal{L}_2$-formulas $\alpha(R)(x_0, \ldots, x_{n-1})$

such that these formulas define a model of $\mathcal{T}_1$ in every model of $\mathcal{T}_2$:

$$\begin{aligned} \mathrm{Mod}(\mathcal{T}_1) &\longleftarrow \mathrm{Mod}(\mathcal{T}_2) \\ (\alpha(R)^{\mathcal{M}})_{R \in \mathcal{L}_1} &\longleftarrow\!\shortmid \mathcal{M}. \end{aligned}$$

# Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$ and $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in WLOG relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping $n$-ary $\mathcal{L}_1$-relns $R \longmapsto \mathcal{L}_2$-formulas $\alpha(R)(x_0, \ldots, x_{n-1})$

such that these formulas define a model of $\mathcal{T}_1$ in every model of $\mathcal{T}_2$:

$$\begin{aligned} \mathrm{Mod}(\mathcal{T}_1) &\longleftarrow \mathrm{Mod}(\mathcal{T}_2) \\ (\alpha(R)^{\mathcal{M}})_{R \in \mathcal{L}_1} &\longleftarrow\!\shortmid \mathcal{M}. \end{aligned}$$

Example Given an $\mathbb{F}_\omega$-action $\mathcal{M} = (C, a_\gamma)_{\gamma \in \mathbb{F}_\omega = \langle g_0, g_1, \ldots \rangle}$, we have the Schreier graph

$$x \; G \; y \;\; :\Longleftrightarrow \;\; \bigvee_n ((a_{g_n}(x) = y) \vee (x = a_{g_n}(y)))\,.$$

# Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$ and $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in WLOG relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping $n\text{-ary } \mathcal{L}_1\text{-relns } R \longmapsto \mathcal{L}_2\text{-formulas } \alpha(R)(x_0, \dots, x_{n-1})$

such that these formulas define a model of $\mathcal{T}_1$ in every model of $\mathcal{T}_2$:

$$\begin{aligned} \mathrm{Mod}(\mathcal{T}_1) &\longleftarrow \mathrm{Mod}(\mathcal{T}_2) \\ (\alpha(R)^{\mathcal{M}})_{R \in \mathcal{L}_1} &\longleftarrow\!\shortmid \mathcal{M}. \end{aligned}$$

Example Given an $\mathbb{F}_\omega$-action $\mathcal{M} = (C, a_\gamma)_{\gamma \in \mathbb{F}_\omega = \langle g_0, g_1, \dots \rangle}$, we have the Schreier graph

$$x \; G \; y \;\; :\Longleftrightarrow \;\; \underbrace{\bigvee_n ((a_{g_n}(x) = y) \vee (x = a_{g_n}(y)))}_{\alpha(G)(x,y)}.$$

This is defined by an interpretation

$\alpha$ : theory of graphs (in $\mathcal{L}_{\mathrm{graph}} = \{G\}$) $\longrightarrow$ theory of $\mathbb{F}_\omega$-actions (in $\mathcal{L}_{\mathbb{F}_\omega} = \{a_\gamma\}_{\gamma \in \mathbb{F}_\omega}$).

# Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$ and $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in WLOG relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping $n\text{-ary } \mathcal{L}_1\text{-relns } R \longmapsto \mathcal{L}_2\text{-formulas } \alpha(R)(x_0, \dots, x_{n-1})$

such that these formulas define a model of $\mathcal{T}_1$ in every model of $\mathcal{T}_2$:

$$\begin{aligned} \mathrm{Mod}(\mathcal{T}_1) &\longleftarrow \mathrm{Mod}(\mathcal{T}_2) \\ (\alpha(R)^{\mathcal{M}})_{R \in \mathcal{L}_1} &\longleftarrow\!\shortmid \mathcal{M}. \end{aligned}$$

Example Given an $\mathbb{F}_\omega$-action $\mathcal{M} = (C, a_\gamma)_{\gamma \in \mathbb{F}_\omega = \langle g_0, g_1, \dots \rangle}$, we have the Schreier graph

$$x \; G \; y \; :\Longleftrightarrow \underbrace{\bigvee_n ((a_{g_n}(x) = y) \vee (x = a_{g_n}(y)))}_{\alpha(G)(x,y)}.$$

This is defined by an interpretation

$\alpha$ : theory of graphs (in $\mathcal{L}_{\text{graph}} = \{G\}$) $\longrightarrow$ theory of $\mathbb{F}_\omega$-actions (in $\mathcal{L}_{\mathbb{F}_\omega} = \{a_\gamma\}_{\gamma \in \mathbb{F}_\omega}$).

Note There is a more general model-theoretic notion of "imaginary interpretation" that we're not using.

# The Feldman–Moore theorem

*Every CBER $E \subseteq X^2$ is generated by a Borel action $\mathbb{Z}_2^{*\omega} \circlearrowright X$.*
*In other words, we may put a transitive $\mathbb{Z}_2^{*\omega}$-action on each $E$-class in a Borel way.*

Proof. By Lusin–Novikov uniformization, $E = \bigcup_{n \in \mathbb{N}} f_n$ for Borel $f_n : X \to X$.
In other words, $C^2 = \bigcup_n f_n$ for $f_n : C \to C$ for each $C \in X/E$ in a Borel way.
Let $f_{ij} := f_i \cap f_j^{-1}$; then $C^2 = \bigcup_{i,j} f_{ij}$ where the $f_{ij}$ are partial bijections.
Since $X$ is standard Borel, there is a family of separating Borel sets $(U_k)_{k \in \mathbb{N}}$. Let

Example The following formulas together yield an interpretation $\mathcal{T}_{\mathbb{Z}_2^{*\omega}} \to$ ???

$$f_{ijk} := f_i \cap f_j^{-1} \cap ((U_k \times X) \triangle (X \times U_k)),$$

i.e., $f_{ijk}(x) = y \;:\Longleftrightarrow\; (f_i(x) = y) \wedge (x = f_j(y)) \wedge (U_k(x) \leftrightarrow \neg U_k(y))$.

Then $C^2 = \bigcup_{i,j,k} f_{ijk} \cup (=_C)$ for partial bijs $f_{ijk}$ w/ disjoint domains & images. So

$$g_{ijk} := f_{ijk} \cup f_{ijk}^{-1} \cup (=_{\neg(\mathrm{dom}(f_{ijk}) \cup \mathrm{im}(f_{ijk}))})$$

i.e., $g_{ijk}(x) = y \;:\Longleftrightarrow\; (f_{ijk}(x) = y) \vee (x = f_{ijk}(y)) \vee (x = y \notin \mathrm{dom}(f_{ijk}) \cup \mathrm{im}(f_{ijk}))$

are countably many involutions covering each $C^2$, defined in a Borel way.
Close under compositions. □

# The Feldman–Moore theorem

**Definition** $\mathcal{T}_{\mathsf{LN}} := \{\forall x, y \bigvee_n (f_n(x) = y) \mid n \in \mathbb{N}\}$ in language $\mathcal{L}_{\mathsf{LN}} := \{f_n\}_{n \in \mathbb{N}}$,

$\mathcal{T}_{\mathsf{sep}} := \{\forall x \neq y \bigvee_k (U_k(x) \leftrightarrow \neg U_k(y))\}$ in language $\mathcal{L}_{\mathsf{sep}} := \{U_k\}_{k \in \mathbb{N}}$.

*Proof.* By Lusin–Novikov uniformization, $E = \bigcup_{n \in \mathbb{N}} f_n$ for Borel $f_n : X \to X$.
In other words, $C^2 = \bigcup_n f_n$ for $f_n : C \to C$ for each $C \in X/E$ in a Borel way.
Let $f_{ij} := f_i \cap f_j^{-1}$; then $C^2 = \bigcup_{i,j} f_{ij}$ where the $f_{ij}$ are partial bijections.
Since $X$ is standard Borel, there is a family of separating Borel sets $(U_k)_{k \in \mathbb{N}}$. Let

**Example** The following formulas together yield an interpretation $\mathcal{T}_{\mathbb{Z}_2^{*\omega}} \to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$

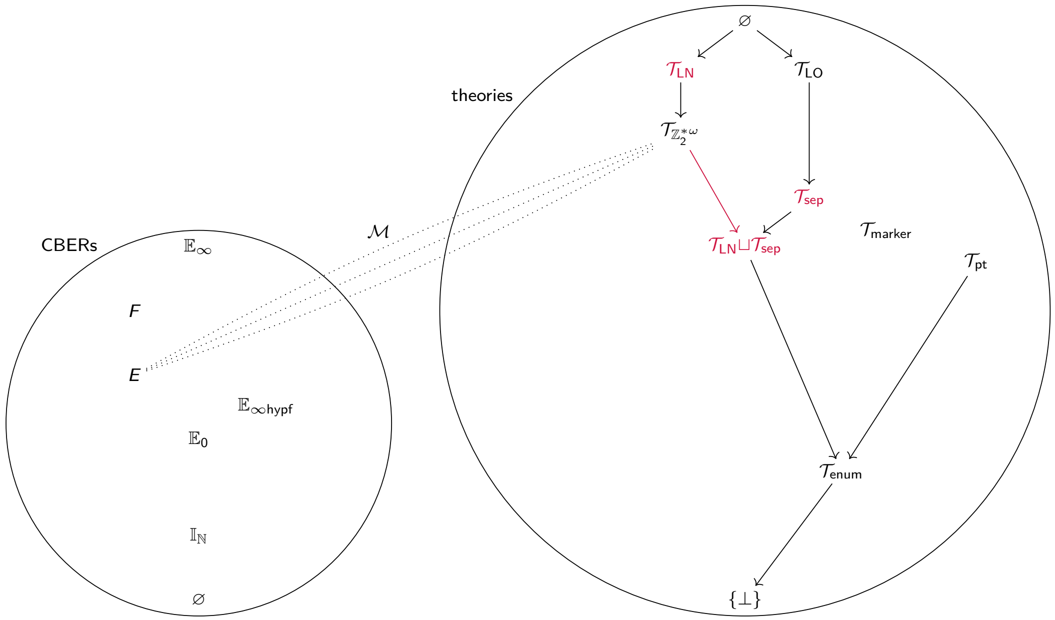
$$\text{i.e., } f_{ijk}(x) = y \;:\Longleftrightarrow\; (f_i(x) = y) \wedge (x = f_j(y)) \wedge (U_k(x) \leftrightarrow \neg U_k(y)).$$

Then $C^2 = \bigcup_{i,j,k} f_{ijk} \cup (=_C)$ for partial bijs $f_{ijk}$ w/ disjoint domains & images. So

$$g_{ijk} := f_{ijk} \cup f_{ijk}^{-1} \cup (=_{\neg(\mathrm{dom}(f_{ijk}) \cup \mathrm{im}(f_{ijk}))})$$

$$\text{i.e., } g_{ijk}(x) = y \;:\Longleftrightarrow\; (f_{ijk}(x) = y) \vee (x = f_{ijk}(y)) \vee (x = y \notin \mathrm{dom}(f_{ijk}) \cup \mathrm{im}(f_{ijk}))$$

are countably many involutions covering each $C^2$, defined in a Borel way.
Close under compositions. □


CBERs
$\mathbb{E}_\infty$
$F$
$E$
$\mathbb{E}_{\infty\mathsf{hypf}}$
$\mathbb{E}_0$
$\mathbb{I}_\mathbb{N}$
$\varnothing$
$\mathcal{M}$
theories
$\varnothing$
$\mathcal{T}_{\mathsf{LN}}$
$\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{marker}}$
$\mathcal{T}_{\mathsf{pt}}$
$\mathcal{T}_{\mathsf{enum}}$
$\{\perp\}$

# CBERs $\hookrightarrow$ theories

**Theorem**

*We have a canonical assignment*

$$\begin{aligned}\{\textit{CBERs}\} &\longhookrightarrow \{\textit{ctbl } \mathcal{L}_{\omega_1\omega} \textit{ theories}\} \\ E &\longmapsto \mathcal{T}_E\end{aligned}$$

*of a theory $\mathcal{T}_E$ to each CBER $E$, called its* ***Scott theory****, such that*

# CBERs $\hookrightarrow$ theories

### Theorem

*We have a canonical assignment*

$$\begin{aligned}\{\textit{CBERs}\} &\longhookrightarrow \{\textit{ctbl } \mathcal{L}_{\omega_1\omega} \textit{ theories}\}\\ E &\longmapsto \mathcal{T}_E\end{aligned}$$

*of a theory* $\mathcal{T}_E$ *to each CBER* $E$*, called its* ***Scott theory****, such that*

(a) *For any other theory* $\mathcal{T}$, $\{\mathcal{T}\text{-structurings of } E\} \cong \{\text{interpretations } \mathcal{T} \to \mathcal{T}_E\}$.

# CBERs $\hookrightarrow$ theories

**Theorem**

*We have a canonical assignment*

$$\begin{aligned}\{\text{CBERs}\} &\hookrightarrow \{\text{ctbl } \mathcal{L}_{\omega_1\omega} \text{ theories}\} \\ E &\longmapsto \mathcal{T}_E\end{aligned}$$

*of a theory $\mathcal{T}_E$ to each CBER $E$, called its **Scott theory**, such that*

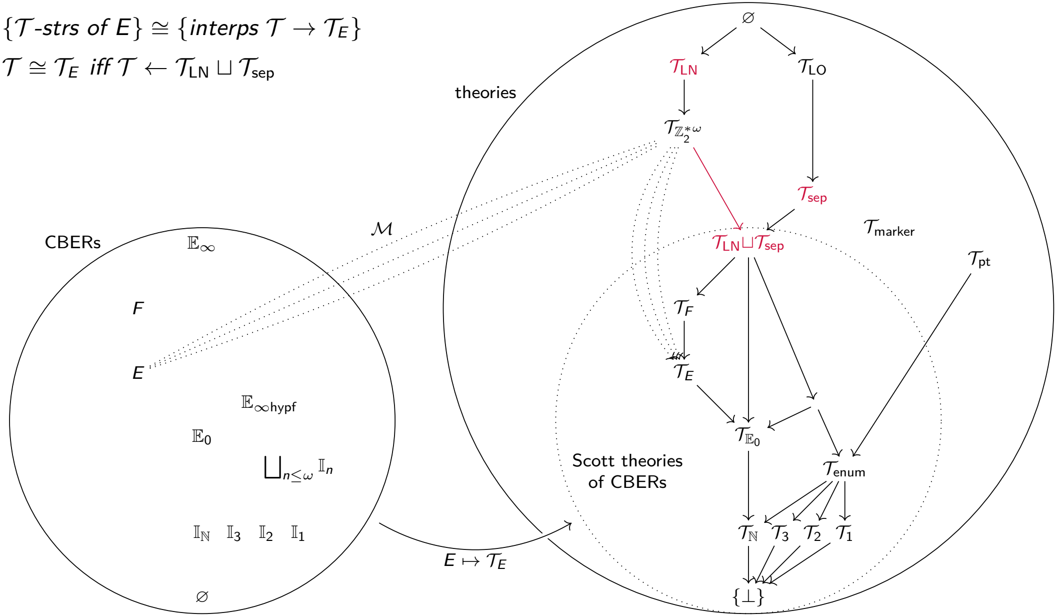
(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

# Main Theorem

(a) $\{\mathcal{T}\text{-strs of } E\} \cong \{\text{interps } \mathcal{T} \to \mathcal{T}_E\}$

(b) $\mathcal{T} \cong \mathcal{T}_E$ iff $\mathcal{T} \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$



# CBERs $\hookrightarrow$ theories

### Theorem

*We have a canonical assignment*

$$\begin{aligned} \{\mathit{CBERs}\} &\longhookrightarrow \{\mathit{ctbl}\ \mathcal{L}_{\omega_1\omega}\ \mathit{theories}\} \\ E &\longmapsto \mathcal{T}_E \end{aligned}$$

*of a theory $\mathcal{T}_E$ to each CBER $E$, called its* ***Scott theory***, *such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

### Corollary (folklore, Banerjee–C.)

*For any theory $\mathcal{T}$, the following are equivalent:*

(a) *Every CBER $E$ is $\mathcal{T}$-structurable.*

(b) *There exists an interpretation $\mathcal{T} \to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

# CBERs $\hookrightarrow$ theories

## Theorem

*We have a canonical assignment*

$$\begin{aligned} \{\mathit{CBERs}\} &\longhookrightarrow \{\mathit{ctbl}\ \mathcal{L}_{\omega_1\omega}\ \mathit{theories}\} \\ E &\longmapsto \mathcal{T}_E \end{aligned}$$

*of a theory $\mathcal{T}_E$ to each CBER $E$, called its **Scott theory**, such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$.*

## Corollary (folklore, Banerjee–C.)

*For any theory $\mathcal{T}$, the following are equivalent:*

(a) *Every $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$ admits $\mathcal{T}_E \leftarrow \mathcal{T}$.*

(b) *There exists an interpretation $\mathcal{T} \to \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$.*

# CBERs $\hookrightarrow$ theories

**Theorem (C.–Kechris 2018, Banerjee–C. 2024)**

*We have a canonical assignment*

$$\{CBERs\} \hookrightarrow \{ctbl\ \mathcal{L}_{\omega_1\omega}\ theories\}$$
$$E \longmapsto \mathcal{T}_E$$

*of a theory $\mathcal{T}_E$ to each CBER $E$, called its* ***Scott theory****, such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}\text{-structurings of } E\} \cong \{\text{interpretations } \mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

**Corollary (folklore, Banerjee–C.)**

*For any theory $\mathcal{T}$, the following are equivalent:*

(a) *Every $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$ admits $\mathcal{T}_E \leftarrow \mathcal{T}$.*

(b) *There exists an interpretation $\mathcal{T} \to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

# CBERs $\hookrightarrow$ theories

**Theorem (C.–Kechris 2018, Banerjee–C. 2024)**

*We have a canonical assignment*

$$\{CBERs\} \longhookrightarrow \{ctbl\ \mathcal{L}_{\omega_1\omega}\ theories\}$$
$$E \longmapsto \mathcal{T}_E$$

*of a theory $\mathcal{T}_E$ to each CBER $E$, called its* ***Scott theory****, such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}\text{-structurings of } E\} \cong \{\text{interpretations } \mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$.*

For a CBER $E \subseteq X^2$, we define $\mathcal{T}_E$ by declaring that models $\mathcal{M} = (Y, \dots)$ of $\mathcal{T}_E$ on a countable set $Y$ to be bijections $Y \to X$ onto an $E$-class.

# CBERs $\hookrightarrow$ theories

**Theorem (C.–Kechris 2018, Banerjee–C. 2024)**

*We have a canonical assignment*

$$\begin{aligned} \{\text{CBERs}\} &\hookrightarrow \{\text{ctbl } \mathcal{L}_{\omega_1\omega} \text{ theories}\} \\ E &\longmapsto \mathcal{T}_E \end{aligned}$$

*of a theory $\mathcal{T}_E$ to each CBER $E$, called its **Scott theory**, such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}\text{-structurings of } E\} \cong \{\text{interpretations } \mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$.*

For a CBER $E \subseteq X^2$, we define $\mathcal{T}_E$ by declaring that models $\mathcal{M} = (Y, \dots)$ of $\mathcal{T}_E$ on a countable set $Y$ to be bijections $Y \to X$ onto an $E$-class.

Given $\mathcal{T} \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$, we have $\mathcal{T} \cong \mathcal{T}_E$ for $E$ on $X = \mathcal{S}_1(\mathcal{T})$, where two 1-types are $E$-related iff they are realized in the same model.

# CBERs $\hookrightarrow$ theories

**Theorem (C.–Kechris 2018, Banerjee–C. 2024)**

*We have a canonical assignment*

$$\begin{aligned} \{\textit{CBERs}\} &\longhookrightarrow \{\textit{ctbl } \mathcal{L}_{\omega_1\omega} \textit{ theories}\} \\ E &\longmapsto \mathcal{T}_E \end{aligned}$$

*of a theory $\mathcal{T}_E$ to each CBER $E$, called its* ***Scott theory***, *such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$.*

For a CBER $E \subseteq X^2$, we define $\mathcal{T}_E$ by declaring that models $\mathcal{M} = (Y, \dots)$ of $\mathcal{T}_E$ on a countable set $Y$ to be bijections $Y \to X$ onto an $E$-class.

Note that for two CBERs $E \subseteq X$ and $F \subseteq Y$,

$$\begin{aligned} \{\text{interps } \mathcal{T}_E \to \mathcal{T}_F\} &\cong \{\mathcal{T}_E\text{-structurings of } F\} \\ &\cong \{\text{Borel families of bijections of each } D \in Y/F \text{ onto an } E\text{-class}\}. \end{aligned}$$

# CBERs $\hookrightarrow$ theories

**Theorem (C.–Kechris 2018, Banerjee–C. 2024)**

*We have a dual equivalence of categories*

$$\begin{aligned} \{CBERs\} &\overset{\cong}{\hookrightarrow} \{ctbl\ \mathcal{L}_{\omega_1\omega}\ theories \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}\} \\ E &\longmapsto \mathcal{T}_E \end{aligned}$$

*of a theory $\mathcal{T}_E$ to each CBER $E$, called its* ***Scott theory***, *such that*

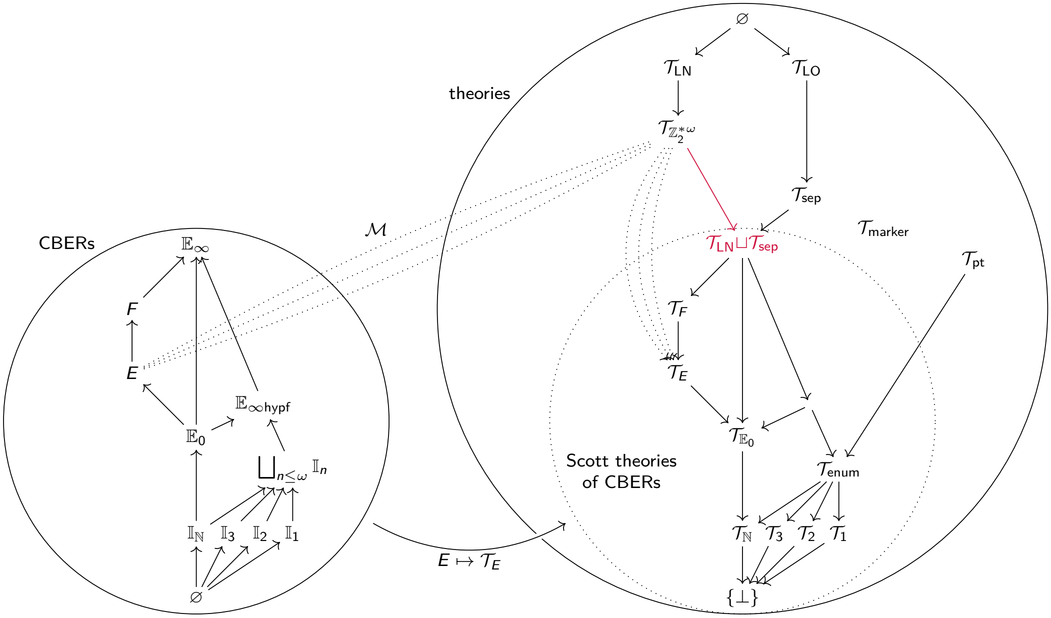
(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$.*

For a CBER $E \subseteq X^2$, we define $\mathcal{T}_E$ by declaring that models $\mathcal{M} = (Y, \dots)$ of $\mathcal{T}_E$ on a countable set $Y$ to be bijections $Y \to X$ onto an $E$-class.

Note that for two CBERs $E \subseteq X$ and $F \subseteq Y$,

$$\begin{aligned} \{\text{interps } \mathcal{T}_E \to \mathcal{T}_F\} &\cong \{\mathcal{T}_E\text{-structurings of } F\} \\ &\cong \{\text{Borel families of bijections of each } D \in Y/F \text{ onto an } E\text{-class}\}. \end{aligned}$$



CBERs
theories
$\mathcal{M}$
$E \mapsto \mathcal{T}_E$
Scott theories of CBERs
$\mathbb{E}_\infty$
$F$
$E$
$\mathbb{E}_{\infty\text{hypf}}$
$\mathbb{E}_0$
$\bigsqcup_{n\le\omega} \mathbb{I}_n$
$\mathbb{I}_\mathbb{N}$ $\mathbb{I}_3$ $\mathbb{I}_2$ $\mathbb{I}_1$
$\varnothing$
$\varnothing$
$\mathcal{T}_{\mathsf{LN}}$ $\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{marker}}$
$\mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{pt}}$
$\mathcal{T}_F$
$\mathcal{T}_E$
$\mathcal{T}_{\mathbb{E}_0}$
$\mathcal{T}_{\mathsf{enum}}$
$\mathcal{T}_\mathbb{N}$ $\mathcal{T}_3$ $\mathcal{T}_2$ $\mathcal{T}_1$
$\{\bot\}$

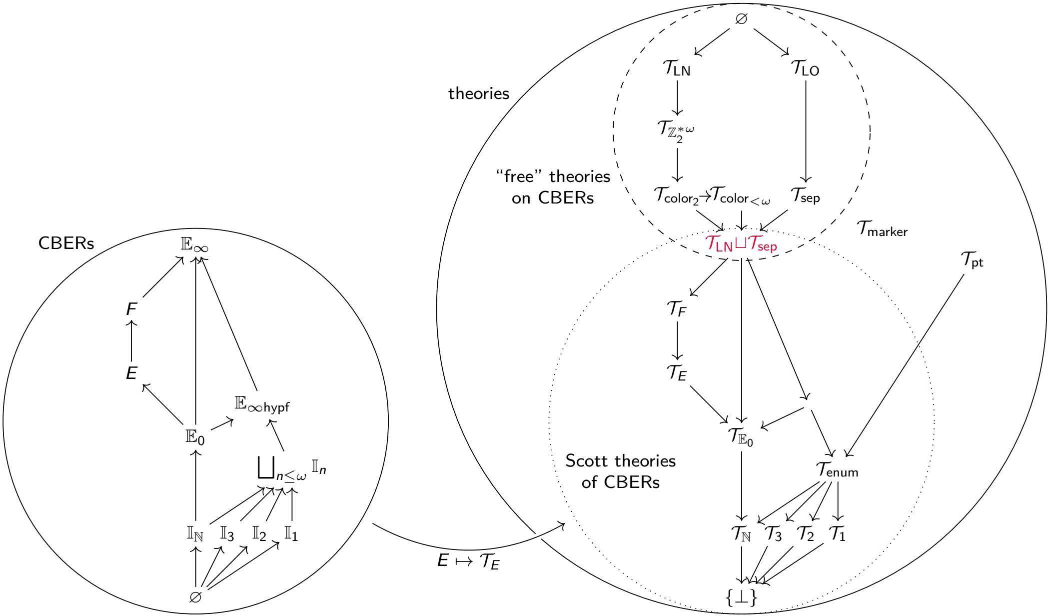
# Free theories

Like Feldman–Moore, many standard Borel constructions amount to interpretations.

Example (Kechris–Miller) Let $\mathcal{T}_{\mathrm{color}_{<\omega}}$ be the theory of $\omega$-colorings of the intersection graph on all finite subsets (in language $\mathcal{L}_{\mathrm{color}_{<\omega}} = \{C_{nk}\}_{n,k\in\omega}$).
Let $\mathcal{T}_{\mathrm{color}_2}$ be the theory of $\omega$-colorings of pairs. There is an interpretation

$$\mathcal{T}_{\mathrm{color}_{<\omega}} \xrightarrow{\text{KM}} \mathcal{T}_{\mathrm{color}_2} \sqcup \mathcal{T}_{\mathrm{LO}} \xrightarrow{\text{FM + lex}} \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}.$$

In other words, every CBER is $\mathcal{T}_{\mathrm{color}_{<\omega}}$-structurable.



CBERs
$\mathbb{E}_\infty$
$F$
$E$
$\mathbb{E}_{\infty\mathsf{hypf}}$
$\mathbb{E}_0$
$\bigsqcup_{n\le\omega}\mathbb{I}_n$
$\mathbb{I}_\mathbb{N}$ $\mathbb{I}_3$ $\mathbb{I}_2$ $\mathbb{I}_1$
$\varnothing$
$E \mapsto \mathcal{T}_E$
theories
"free" theories on CBERs
Scott theories of CBERs
$\varnothing$
$\mathcal{T}_{\mathsf{LN}}$
$\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{color}_2}$
$\mathcal{T}_{\mathsf{color}_{<\omega}}$
$\mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{marker}}$
$\mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{pt}}$
$\mathcal{T}_F$
$\mathcal{T}_E$
$\mathcal{T}_{\mathbb{E}_0}$
$\mathcal{T}_{\mathsf{enum}}$
$\mathcal{T}_\mathbb{N}$ $\mathcal{T}_3$ $\mathcal{T}_2$ $\mathcal{T}_1$
$\{\bot\}$

# Free theories

Like Feldman–Moore, many standard Borel constructions amount to interpretations.

Example (Kechris–Miller) Let $\mathcal{T}_{\mathsf{color}_{<\omega}}$ be the theory of $\omega$-colorings of the intersection graph on all finite subsets (in language $\mathcal{L}_{\mathsf{color}_{<\omega}} = \{C_{nk}\}_{n,k\in\omega}$).
Let $\mathcal{T}_{\mathsf{color}_2}$ be the theory of $\omega$-colorings of pairs. There is an interpretation

$$\mathcal{T}_{\mathsf{color}_{<\omega}} \xrightarrow{\text{KM}} \mathcal{T}_{\mathsf{color}_2} \sqcup \mathcal{T}_{\mathsf{LO}} \xrightarrow{\text{FM + lex}} \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}.$$

In other words, every CBER is $\mathcal{T}_{\mathsf{color}_{<\omega}}$-structurable.

We give a detailed analysis of how much is needed to perform such constructions.

Example (Banerjee-C.) None of the interpretabilities

$$\mathcal{T}_{\mathsf{LN}} \longrightarrow \mathcal{T}_{\mathbb{Z}_2^{*\omega}} \longrightarrow \mathcal{T}_{\mathsf{color}_2} \longrightarrow \mathcal{T}_{\mathsf{color}_{<\omega}} \longrightarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$$

can be reversed.

# Free theories

Like Feldman–Moore, many standard Borel constructions amount to interpretations.

Example (Kechris–Miller) Let $\mathcal{T}_{\text{color}_{<\omega}}$ be the theory of $\omega$-colorings of the intersection graph on all finite subsets (in language $\mathcal{L}_{\text{color}_{<\omega}} = \{C_{nk}\}_{n,k\in\omega}$).
Let $\mathcal{T}_{\text{color}_2}$ be the theory of $\omega$-colorings of pairs. There is an interpretation

$$\mathcal{T}_{\text{color}_{<\omega}} \xrightarrow{\text{KM}} \mathcal{T}_{\text{color}_2} \sqcup \mathcal{T}_{\text{LO}} \xrightarrow{\text{FM + lex}} \mathcal{T}_{\text{LN}} \sqcup \mathcal{T}_{\text{sep}}.$$

In other words, every CBER is $\mathcal{T}_{\text{color}_{<\omega}}$-structurable.

We give a detailed analysis of how much is needed to perform such constructions.

Example (Banerjee-C.) None of the interpretabilities

$$\underbrace{\mathcal{T}_{\mathbb{F}_\omega} \rightleftarrows \mathcal{T}_{\text{LN}} \longrightarrow \mathcal{T}_{\mathbb{Z}_2^{*\omega}} \longrightarrow \mathcal{T}_{\text{color}_2}}_{\text{increasingly strong versions of FM}} \longrightarrow \mathcal{T}_{\text{color}_{<\omega}} \overset{\text{unless size } = 2}{\underset{\longrightarrow}{\longleftarrow}} \mathcal{T}_{\text{LN}} \sqcup \mathcal{T}_{\text{sep}}$$

$$\mathcal{T}_{\text{color}_{<\omega}} \upharpoonleft\!\downharpoonright \mathcal{T}_{\text{color}_{\leq 3}}$$

can be reversed. However, these can be.

# Free theories

Like Feldman–Moore, many standard Borel constructions amount to interpretations.

Example (Kechris–Miller) Let $\mathcal{T}_{\mathsf{color}_{<\omega}}$ be the theory of $\omega$-colorings of the intersection graph on all finite subsets (in language $\mathcal{L}_{\mathsf{color}_{<\omega}} = \{C_{nk}\}_{n,k\in\omega}$).
Let $\mathcal{T}_{\mathsf{color}_2}$ be the theory of $\omega$-colorings of pairs. There is an interpretation

$$\mathcal{T}_{\mathsf{color}_{<\omega}} \xrightarrow{\mathsf{KM}} \mathcal{T}_{\mathsf{color}_2} \sqcup \mathcal{T}_{\mathsf{LO}} \xrightarrow{\mathsf{FM}\,+\,\mathsf{lex}} \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}.$$

In other words, every CBER is $\mathcal{T}_{\mathsf{color}_{<\omega}}$-structurable.

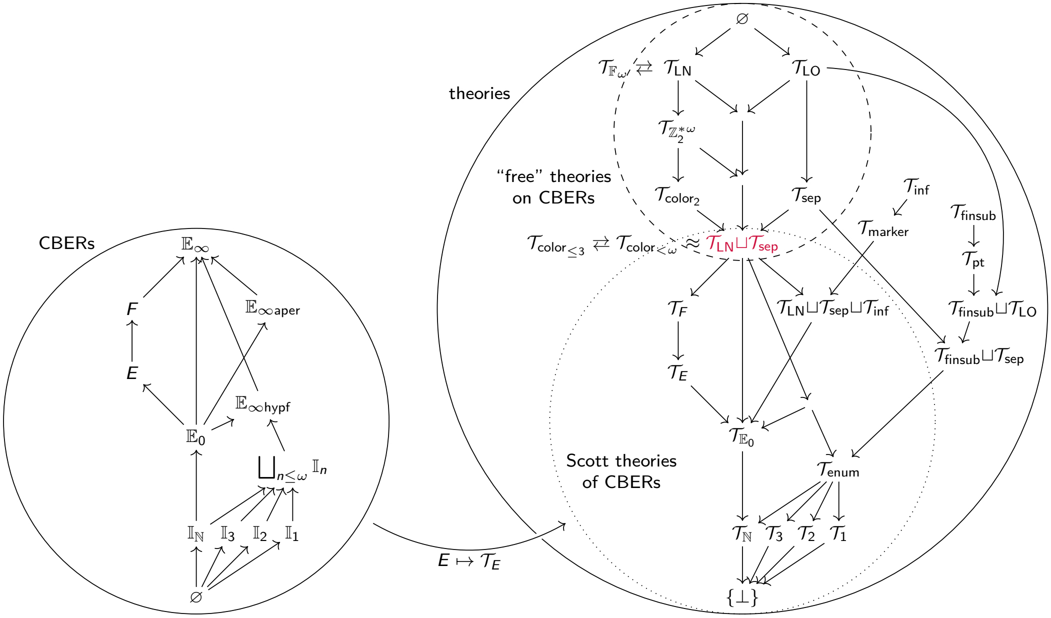
We give a detailed analysis of how much is needed to perform such constructions.

Example (Banerjee-C.) None of the interpretabilities

$$\underbrace{\mathcal{T}_{\mathbb{F}_\omega} \rightleftarrows \mathcal{T}_{\mathsf{LN}} \longrightarrow \mathcal{T}_{\mathbb{Z}_2^{*\omega}} \longrightarrow \mathcal{T}_{\mathsf{color}_2}}_{\text{increasingly strong versions of FM}} \longrightarrow \mathcal{T}_{\mathsf{color}_{<\omega}} \overset{\text{unless size} = 2}{\overset{\leftarrow}{\longrightarrow}} \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$$

$$\mathcal{T}_{\mathsf{color}_{<\omega}} \upharpoonleft\!\downharpoonright \mathcal{T}_{\mathsf{color}_{\le 3}}$$

can be reversed. However, these can be.
Proofs of $\not\leftarrow$: e.g., $(\mathbb{Z}, (-) + n)_{n\in\mathbb{Z}} \models \mathcal{T}_{\mathsf{LN}}$, but has nontrivial automorphisms.


CBERs
$\mathbb{E}_\infty$
$F$
$\mathbb{E}_{\infty\mathsf{aper}}$
$E$
$\mathbb{E}_{\infty\mathsf{hypf}}$
$\mathbb{E}_0$
$\bigsqcup_{n\le\omega}\mathbb{I}_n$
$\mathbb{I}_\mathbb{N}$ $\mathbb{I}_3$ $\mathbb{I}_2$ $\mathbb{I}_1$
$\varnothing$
$E \mapsto \mathcal{T}_E$
theories
"free" theories on CBERs
Scott theories of CBERs
$\varnothing$
$\mathcal{T}_{\mathbb{F}_\omega} \rightleftarrows \mathcal{T}_{\mathsf{LN}}$
$\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{color}_2}$
$\mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{inf}}$
$\mathcal{T}_{\mathsf{finsub}}$
$\mathcal{T}_{\mathsf{marker}}$
$\mathcal{T}_{\mathsf{color}_{\le 3}} \rightleftarrows \mathcal{T}_{\mathsf{color}_{<\omega}} \approx \mathcal{T}_{\mathsf{LN}}\sqcup\mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{pt}}$
$\mathcal{T}_F$
$\mathcal{T}_{\mathsf{LN}}\sqcup\mathcal{T}_{\mathsf{sep}}\sqcup\mathcal{T}_{\mathsf{inf}}$
$\mathcal{T}_{\mathsf{finsub}}\sqcup\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_E$
$\mathcal{T}_{\mathsf{finsub}}\sqcup\mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathbb{E}_0}$
$\mathcal{T}_{\mathsf{enum}}$
$\mathcal{T}_\mathbb{N}$ $\mathcal{T}_3$ $\mathcal{T}_2$ $\mathcal{T}_1$
$\{\bot\}$