A hierarchy of CLI Polish groups and their dynamics

Joint work with Aristotelis Panagiotopoulos

Given e.r.s E and F living on Polish spaces X and Y resp, a **Borel reduction** (X, E) to (Y, F) is a Borel function $f : X \to Y$ s.t. $x E y$ iff $f(x) F f(y)$ for every $x, y \in X$.

A Polish group G **classifies** (X, E) if there is a continuous action $\boxed{G \curvearrowright Y}$ on Polish Y and a Borel reduction of (X, E) to (Y, E^G_Y).

Say G has stronger classification strength than H if every (X, E) classifiable by H is classifiable by G.

For a class $\mathcal{C}$ of Polish groups, we say $G \in \mathcal{C}$ has maximal classification strength in $\mathcal{C}$ if it has stronger c.s. than all groups in $\mathcal{C}$.

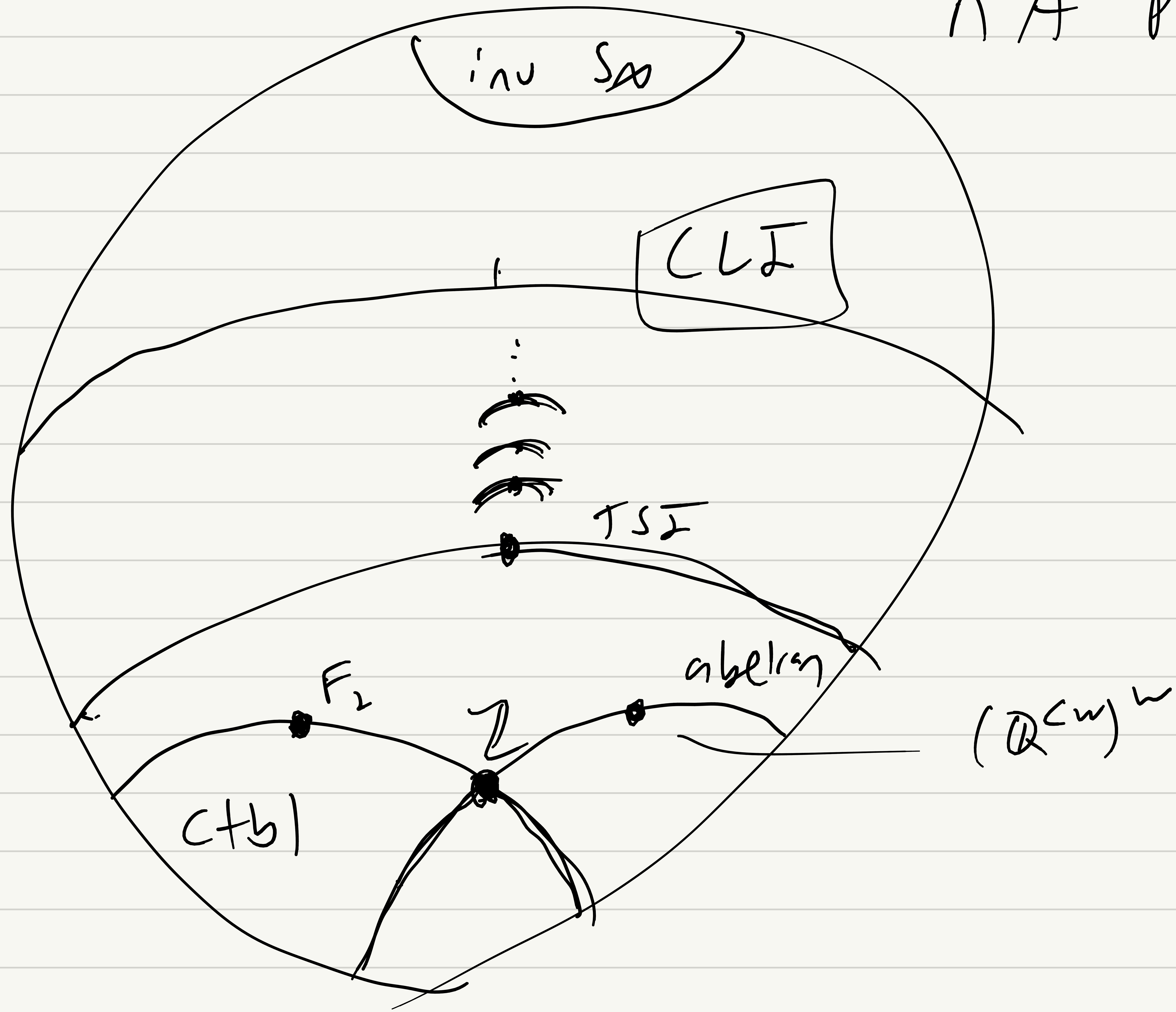
We say G involves H if there is a closed subgroup $G_0 \leq G$ and a continuous surjective hom from G_0 to H.

Thm (Mackey, Hjorth) $\boxed{\text{involves} \Rightarrow \text{stronger c.s.}}$

<u>Ex</u> $\mathrm{Aut}(\mathbb{Q}, <)$ involves S_∞

So $\mathrm{Aut}(\mathbb{Q})$ and S_∞ have <u>maximal c.s.</u> in class of nA Polish groups.

$\cap$ A Polish groups



A metric d on G is <u>left-invariant</u> if $d(g,g') = d(hg,hg')$ for all $h,g,g' \in G$.

<u>Thm (Birkhoff–Kakutani)</u> Every Polish group has a compatible left-inv metric.

CLI $\Rightarrow$ complete compatible left-inv metric

$\Leftrightarrow$ all comp. left-inv metrics are complete

<u>Prop</u> d_e left-inv. (g_n) is Cauchy wrt d_e iff

$\forall V \subseteq_1 G$ open, $\exists N$ $\forall n \geq N$, $g_n^{-1} g_N \in V$] (g_n) is left-Cauchy

Pf $d_e(g_n, g_N) = d(g_N g_n, 1)$

M ctbl structure

Ex Natural left-inv metric on $\mathrm{Aut}(M)$

$$d_\ell(g,h) = \sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right)^{g(n)\neq h(n)}$$

(g_n) is left-Cauchy iff converges to a $L_{\omega_1\omega}$ embedding

Thm (Gao) $G = \mathrm{Aut}(M)$ TFAE

(1) G cli

(2) all $L_{\omega_1\omega}$ elem embeddings are surjective

(3)

Thm (Deissler) Add

(4) $\mathrm{Drk}(a,\emptyset) < \infty$ for all $a \in M$

(5) $\mathrm{Drk}(a,\emptyset) < \omega_1$ for all $a \in M$

Deissler Rank Given $a \in M$, $\bar{b} \in M^{<\omega}$, define

$\mathrm{Drk}(a, \bar{b})$ recursively:

(1) $\mathrm{Drk}(a, \bar{b}) \leq 0$ iff $\forall \pi \in \mathrm{Aut}(M)$ if $\pi(\bar{b}) = \bar{b}$ then $\pi(a) = a$

(2) $\mathrm{Drk}(a, \bar{b}) \leq \alpha$ iff $\exists \bar{c} \in M^{<\omega}$ s.t. $\forall \pi \in \mathrm{Aut}(M)$ s.t. $\pi(\bar{b}) = \bar{b}$, $\mathrm{Drk}(a, \bar{b}\pi(\bar{c})) < \alpha$.

For general Polish groups G general Polish group, $U, V \leq_o G$

define $\mathrm{rk}(U, V)$ recursively

(1) $\mathrm{rk}(U, V) \leq 0$ iff $V \subseteq U$

(2) $\mathrm{rk}(U, V) \leq \alpha$ iff $\exists W \leq_o G$ s.t. $\forall v \in V$, $\mathrm{rk}(U, vWv^{-1}) < \alpha$

[Joint work with Aristoteles Panagiotopoulos]

Thm (A.-Panagiotopoulos) TFAE

(1) G cli

(2) every left-Cauchy is right-Cauchy

(3) $rk(U, G) < \infty$ for every $U \subseteq_1 G$

(4) $rk(U, G) < \omega_1$ for every $U \subseteq_1 G$

Pf (1) $\Rightarrow$ (2) immediate

(2) $\Rightarrow$ (1) $D(g,h) = \underline{d_e(g,h)} + \underline{d_e(g^{-1},h^{-1})}$ $\leftarrow$ always complete

$(3) \Rightarrow (2)$ Sps $rk(\mathcal{U}, G) < \infty$ for every $\mathcal{U} \leq_1 G$

Sps (g_n) is a left-Cauchy seq.

__Claim__ Sps $\mathcal{U}, \mathcal{V} \leq_1 G$ $rk(\mathcal{U}, \mathcal{V}) \leq \alpha$ and there is some $m_0 \in \mathbb{N}$ s.t. $\forall n \geq m_0$, $g_n \in \mathcal{V} g_{m_0}$. Then there exist $n_0 \in \mathbb{N}$ s.t. $g_n \in \mathcal{U} g_{n_0}$ for all $n \geq n_0$

__Pf__ $\boxed{\text{Induct on } \alpha \text{ for all } \mathcal{V}}$

$(\alpha = 0)$ $rk(\mathcal{U}, \mathcal{V}) \leq 0 \Rightarrow \mathcal{V} \leq \mathcal{U}$ Let $n_0 = m_0$

$(\alpha > 0)$ Sps $rk(\mathcal{U}, \mathcal{V}) \leq \alpha$ and claim true below α

Fix $\mathcal{W} \leq_1 G$ s.t. $rk(\mathcal{U}, v\mathcal{W}v^{-1}) < \alpha$ for all $v \in \mathcal{V}$

By left-Cauchyness of (g_n) find some $m_1 \geq m_0$

s.t. $\underbrace{g_n \in g_{m_1}(g_{m_0}^{-1} V g_{m_0})}$ for all $n \geq m_1$,

Find $v \in V$ s.t. $\underline{g_{m_1} = v g_{m_0}}$

OBS for all $n \geq m_1$ we have

$$g_n \in \boxed{g_{m_1}}(g_{m_0}^{-1} W \boxed{g_{m_0}}) = v g_{m_0} g_{m_0}^{-1} W v^{-1} g_{m_1}$$

$$= (v W v^{-1}) g_{m_1}$$

By IH, find $n_0 \geq m_1$ s.t. $g_n \in U g_{n_0}$ for all $n \geq n_0$

Lemma If $rk(\mathcal{U}, \mathcal{V}) = \infty$ then for any W there is some $v \in \mathcal{V}$ s.t. $rk(\mathcal{U}, vWv^{-1}) = \infty$.

For every ord α $\Rightarrow$ $\boxed{\alpha^*}$ scattered linear order

Hausdorff rank α

Dessler rank α

$(L, <)$, L^*: l.o. on functions from $L \to \mathbb{Z}$ where

$f <^* g$ iff $f(a) < g(a)$ for the <u>greatest</u> $a \in L$

s.t. $f(a) \neq g(a)$

L is not a well-order $\Rightarrow$ <u>$\mathrm{Aut}(L^*)$ in</u> S_∞

not CLI

Thm (A. - Panagiotopoulos) The class of CLI Polish groups is Π^1_1-complete

We say G is <u>α-balanced</u> iff $rk(U, G) < \alpha$ for every $U \subseteq_1 G_1$

<u>Thm</u> (A. - Panagiotopoulos) For every α there is an α-balanced Polish group G and a continuous action $G \curvearrowright X$ that is not classifiable by a β-balanced Polish group for any $\beta < \alpha$.

Given structure M, the generalized Bernoulli shift of $\mathrm{Aut}(M)$ is the shift action $\mathrm{Aut}(M) \curvearrowright (2^\omega)^M$

<u>Thm</u> The generalized Bernoulli shift of $\boxed{\mathrm{Aut}(\alpha^*)}$ is <u>generically α-unbalanced</u> and thus not classifiable by α-balanced Polish group.

Generally unbalanced

$$\underline{\leftrightsquigarrow^1_1} \supseteq \underline{\sim^2_1} \supseteq \dots \supseteq \boxed{\leftrightsquigarrow^\alpha_1} \supseteq \boxed{\sim_1} \supseteq \boxed{\in^G_X}$$

If G α-balanced $\boxed{\leftrightsquigarrow^\alpha_1} = \boxed{\sim_1}$

$\boxed{\text{"}\leftrightsquigarrow^1_\alpha \text{ strongly connected"}}$ $\quad \boxed{\sim_\alpha}$

$$\mathrm{rk}(G) = \sup\{\mathrm{rk}(U,G)+1 \mid U \subseteq_1 G\}$$

$\mathrm{rk}(U,G)$ finite, unbounded