Construction and Obstruction Results in Baire Measurable Combinatorics

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Descriptive Combinatorics / Notation

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We call such colorings **Baire measurable** or μ-**measurable**.

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For any Borel probability measure ν on X, the free part of the shift preserves the product measure $\mu = \nu^{|\Gamma|}$.

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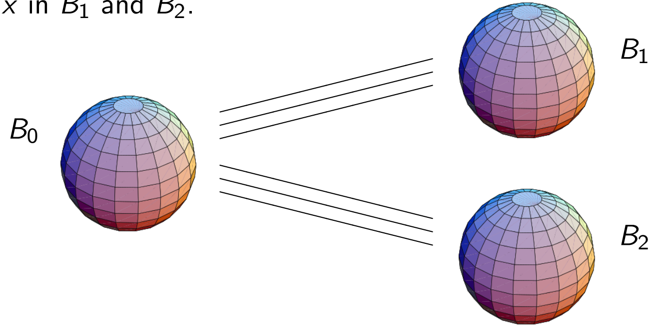

- A perfect matching in G is an instance of the Banach Tarski paradox.

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If G is a locally finite Borel graph and $h : \mathbb{N} \to \mathbb{N}$ is a function, then there is a Baire measurable decomposition of the vertices

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such that each A_n is a set of pairwise distances $> h(n)$.

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- For fast enough growing h, this decomposition can produce Baire measurable toast in a bounded degree Borel graph.
- If G is a measure preserving graph and h grows fast enough, such a decomposition cannot be done measurably.

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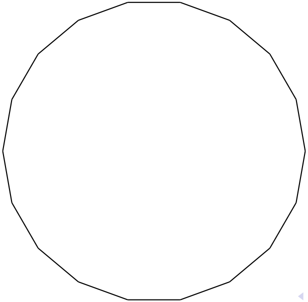

Descriptive Complexity Classes

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Given a group Γ, a **locally checkable labeling** (LCL) problem

$$\Pi = (W, \Lambda, A)$$

consists of a finite set of generators $W \subseteq \Lambda$, a finite set of labels Λ, and a finite set of allowed configurations $A \subseteq \Lambda^W$.

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E.g. matching, k-coloring for fixed k, etc.

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We can also define classes FIID(Γ), FFIID(Γ), CONT(Γ), etc.

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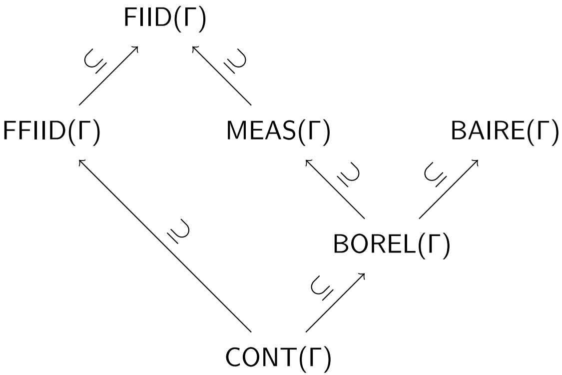

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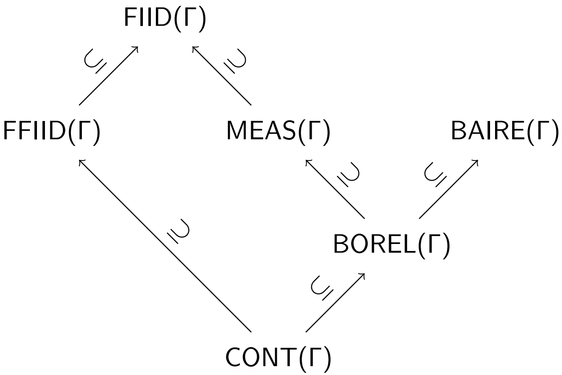

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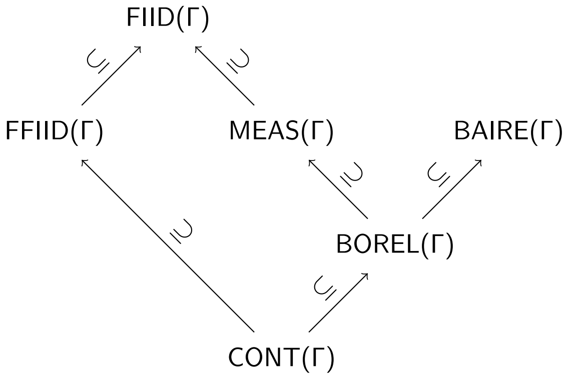

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Brandt, Chang, Grebík, Grunau, Rozhoň, and Vidnyánszky showed

$$\text{MEAS}(\mathbb{F}_n) \subseteq \text{BAIRE}(\mathbb{F}_n)$$

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For $q \in \mathbb{N}$ a q-**toast** is a Borel family $\mathcal{T} \subseteq [X]^{<\infty}$ of finite sets such that

- for all $K, L \in \mathcal{T}$
$$K \cap L = \emptyset \text{ or } K \subseteq L \text{ or } L \subseteq K$$
- for distinct $K, L \in \mathcal{T}$
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We say $\mathcal{T}$ is **complete** if $\bigcup \mathcal{T} = X$.

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Theorem (Gao, Jackson, Krohne, Seward)

Free Borel actions of $\mathbb{Z}^n$ admit complete q-toast.

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For any free Borel action of $\mathbb{Z}^n \curvearrowright X$ and any Borel probability measure μ, X admits a q-toast $\mathcal{T}$ with all pieces squares and $\mu(\bigcup \mathcal{T}) = 1$.

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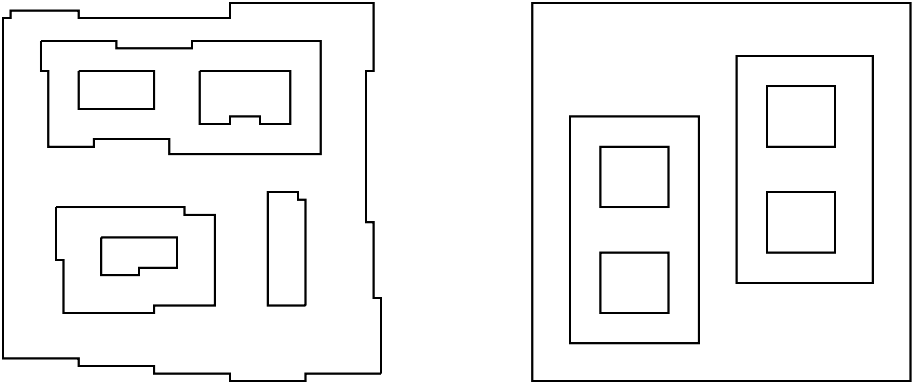

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Step 3: Repeat step 2 at larger scales and use Borel-Cantelli.

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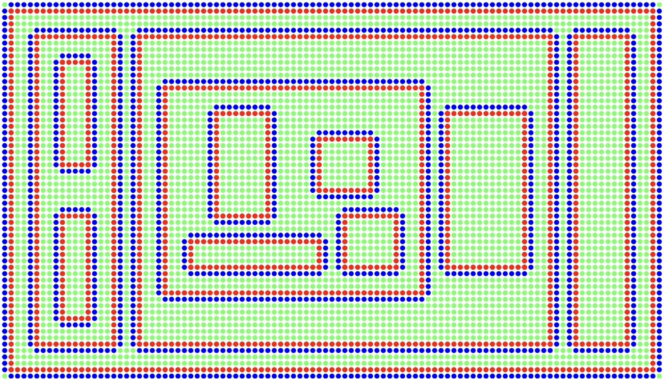

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- Also Q is comeager in a basic open set U, and finitely many translates of U cover X, by compactness and the density of orbits.
- So finitely many translates of Q by powers of e_1 cover a comeager subset of X.
- This places an upper bound on the sidelengths of the boundary rectangles in the solution to Π.

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- Π admits Baire measurable solutions on $F(\{\pm e_i\}, X)$
- $\Pi \notin \text{FFIID}(\mathbb{Z}^n)$
- $\Pi \in \text{COMPUTABLE}(\mathbb{Z}^n)$

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- $\Pi \notin \mathrm{FFIID}(\mathbb{Z}^n)$
- $\Pi \in \mathrm{COMPUTABLE}(\mathbb{Z}^n)$

Theorem (Berlow, Bernshteyn, L., Weilacher)

Fix $n \geq 2$. There is an LCL Π' on $\mathbb{Z}^n$ such that

- $\Pi \notin \mathrm{FIID}(\mathbb{Z}^n)$
- Π admits Baire measurable solutions on $F(\{\pm e_i\}, X)$
- $\Pi \in \mathrm{COMPUTABLE}(\mathbb{Z}^n)$

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Problem

Can we describe BAIRE($\mathbb{Z}^n$)? Is it computable?

Thank You