Rutgers
The State University of New Jersey

Idealistic equivalence relations remastered

joint with L. Motto Ros (Torino)

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Aknowledgements

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Classical Descriptive set theory: definable subsets of Polish spaces ($\mathbb{R}$, 2^ω, ω^ω, etc...)

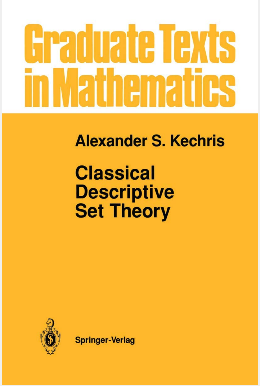

Figure: Kechris' book.

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- $E \sim_B F$ if and only if $E \leq_B F$ and $F \leq_B E$.

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Here $S_\infty = \{f : \mathbb{N} \xrightarrow[\text{su}]{1-1} \mathbb{N}\}$ acts on $X_{\mathcal{L}}$ by permuting the universe set.

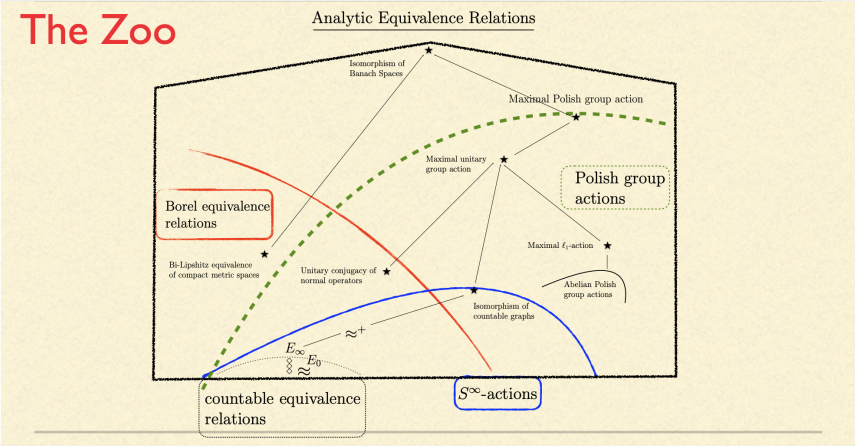

Figure: Courtesy of Matt Foreman.

Idealistic equivalence relations

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An equivalence relation E on a Polish space X is **idealistic** if there is a map assigning to each $C \in X/E$ a nontrivial σ-ideal I_C on C such that:

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This is a technical definition that is better motivated by examples.

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  $$A \in I_C \iff \mu_x(A) = 0.$$
- Let $E = E^X_G$ for some continuous action $G \curvearrowright X$.For any $x \in X$, define the corresponding ideal $I_{[x]_E}$ by
  $$A \in I_{[x]_E} \iff \{g \colon g \cdot x \in A\} \in \mathrm{MGR}(G).$$

Idealistic equivalence relations (counterexample)

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Theorem (Kechris-Louveau '97)

Let E be a ***non-smooth***, ***hypersmooth*** *Borel equivalence relation. Then exactly one of the following holds:*

1. $E \sim_B E_0$,
2. $E \sim_B E_1$.

Counterexample

Theorem (Kechris–Louveau'97)

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Theorem (Kechris–Louveau'97)

Let E be an orbit equivalence relation, then $E_1 \not\leq_B E$.

This is commonly used to prove that certain equivalence relations are not classifiable by orbit equivalence relations.

Question (Kechris–Louveau '97)

If E is a Borel equivalence relation, is it true that either

1. $E_1 \leq_B E$, or
2. E is idealistic?

Hoping for a dichotomy

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E_1 dichotomy conjecture (Hjorth–Kechris '97)

Yes.

Intermezzo

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While $E \leq_B F$ and $F \leq_B E$ does not imply $E \cong_B F$, we have

$$E \sqsubseteq_{cB} F \text{ and } F \sqsubseteq_{cB} E \iff E \cong_{cB} F.$$

For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ define

$$(x_1, y_1)\ E\ (x_2, y_2) \quad \iff \quad x_1 = x_2.$$

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So $\cong_{\mathsf{GRAPHS}} \sim_B \cong_{\mathsf{TREE}}$ but $\cong_{\mathsf{GRAPHS}} \not\simeq_{cB} \cong_{\mathsf{TREE}}$.

End Intermezzo

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Theorem (Hjorth '05)

*There is a **Borel equivalence relation** R such that:*

1. $R \leq_B F$ *for some **countable Borel equivalence relation*** F
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Theorem (Kechris–MacDonald '16)

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- *Hjorth's R is not idealistic and E_1 **dichotomy is false**.*
- *The class of idealistic equivalence relations is not closed downward.*

«So the logicians entered the picture in their usual style, as spoilers.»

(Moschovakis)

Theorem (Becker 2001)

Assume $\mathbf{\Sigma}^1_1$ determinacy. There is an equivalence relation $E_{\mathbb{B}}$ on a Polish space $\mathbb{X}$ such that

1. *$E_{\mathbb{B}}$ is $\mathbf{\Sigma}^1_1$;*
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This answered a question of Kechris, who previously asked whether (1)–(3) implies ¬(4).

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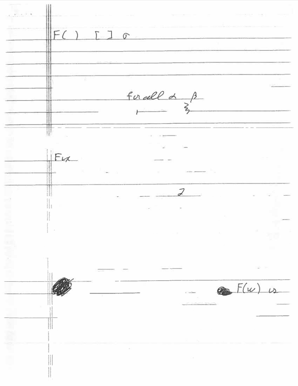

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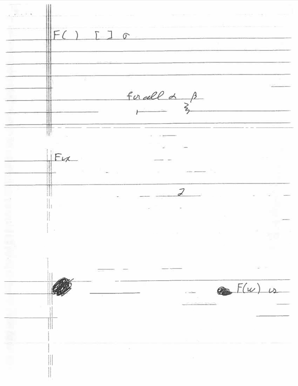

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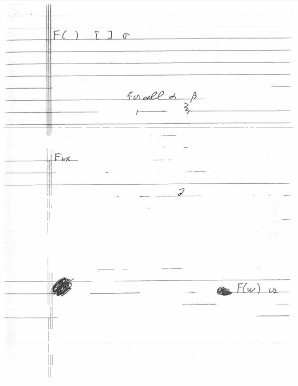

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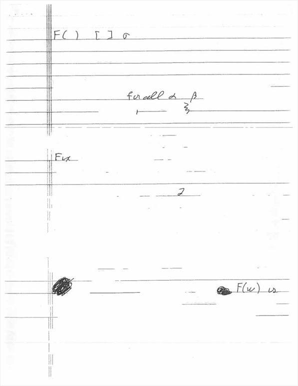

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Orbit vs. Idealistic (remastered)

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Moreover, let $\mathcal{I}$ be the class of $\mathbf{\Sigma}^1_1$ equivalence relations with (1)–(4').

Theorem (Motto Ros-C. 2025)

Assume $\mathbf{\Sigma}^1_1$ determinacy. The poset $(\mathcal{P}(\omega)/fin, \subseteq)$ embeds into $(\mathcal{I}, \sqsubseteq_{cB})$.

Abelian p-groups

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- $C_p \oplus C_{p^2} \oplus \cdots \oplus C_{p^n} \oplus \cdots$
- The quasi-cyclic p-group $\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p]/\mathbb{Z}$

Any countable abelian p-group G decomposes as

$$G = D(G) \oplus R(G)$$

The **divisible part** $D(G) = \underbrace{\mathbb{Z}(p^\infty) \oplus \cdots \oplus \mathbb{Z}(p^\infty)}_{r \text{ times}}$ for $r = 0, 1, \ldots, \omega$.

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The **reduced part** $R(G)$ is completely classified by the **Ulm invariant**, which is a sequence in $(\mathbb{N} \cup \{\infty\})^{<\omega_1}$ that completely encodes the isomorphism type of $R(G)$.

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For $\mathcal{A}, \mathcal{B} \in X_T$ define

$$(\mathcal{A}, \mathcal{B}) \in E_{\mathbb{B}} \qquad \iff \qquad H(\mathcal{A}) \cong_{\mathcal{L}'} H(\mathcal{B}),$$

where $H(\mathcal{A})$ and $H(\mathcal{B})$ are the $\mathcal{L}'$-reducts of $\mathcal{A}$ and $\mathcal{B}$, respectively.

Definition of $E_{\mathbb{B}}$ (cont'd)

Note that if

$$\begin{aligned} \mathcal{A} =& R(\mathcal{A}) \oplus \mathbb{Z}(p^\infty)^\omega \oplus \underbrace{\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)}_{\text{unnamed}} \\ \mathcal{B} =& R(\mathcal{A}) \oplus \mathbb{Z}(p^\infty)^\omega \oplus \underbrace{\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)}_{\text{unnamed}}, \end{aligned}$$

then $\mathcal{A} \not\cong_{\mathcal{L}} \mathcal{B}$ but $(\mathcal{A}, \mathcal{B}) \in E_{\mathbb{B}}$.

Theorem (Becker 2001)

3. $E_{\mathbb{B}}$ ***is idealistic.***

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Definition

Suppose that $E \subseteq F$ are analytic equivalence relations on X. If $\theta \colon X \to X$ is a homomorphism from F to E such that $\theta(x)\ F\ x$ for all $x \in X$, then we say that θ **selects an E-class within each F-class**.

A few comments on the proof (cont'd)

Proposition

Let E be an orbit equivalence relation induced by a Borel action $G \curvearrowright X$ of a Polish group G on a Polish space E. Let $F \supseteq E$ be any equivalence relation on X.

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Let E be an orbit equivalence relation induced by a Borel action $G \curvearrowright X$ of a Polish group G on a Polish space E. Let $F \supseteq E$ be any equivalence relation on X. If there is a Borel map $\theta\colon X \to X$ selecting an E-class within every F-class, then F is idealistic.

In our case let

$$\theta(\mathcal{A}) = \mathcal{A} \oplus \underbrace{\mathbb{Z}(p^\infty) \oplus \cdots \oplus \mathbb{Z}(p^\infty) \oplus \cdots}_{\omega \text{ unnamed copies}}$$

Question (Becker)

Is Becker's equivalence relation $E_{\mathbb{B}}$ **Borel bi-reducible** with an orbit equivalence relation? It is not hard to see that $E_{\mathbb{B}}$ is **Borel reducible** to $\cong_{\mathcal{L}'}$ by definition.

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Question (Becker)

Can we remove the hypothesis of $\mathbf{\Sigma}^1_1$-determinacy?

New E_1 Conjecture

Let E be a Borel equivalence relation. Then either $E_1 \leq_B E$ or E is Borel reducible to an idealistic (or orbit) equivalence relation.

More about orbit vs. idealistic

Question

Is every idealistic equivalence relation on a Polish space Borel bi-reducible to an orbit equivalence relation?

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Is every idealistic equivalence relation on a Polish space Borel bi-reducible to an orbit equivalence relation?

Proposition

Let E be an idealistic equivalence relation, and suppose that $E \leq_B F$ for some ***Borel orbit*** *equivalence relation F. Then E is* ***classwise Borel isomorphic*** *to (and hence Borel bireducible with) a Borel orbit equivalence relation.*

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Thank You!