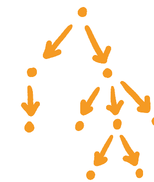
Hyperfinite Partial Orders



joint work with
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Caltech Logic Seminar 2026

① Introduction

Motivating question Is there a Borel function $F: 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $x, y \in 2^{\mathbb{N}}$, $x <_T y \Rightarrow F(x) <^* F(y)$?

Definition For $f, g \in \mathbb{N}^{\mathbb{N}}$, $f <^* g$ means $\exists N \forall n \geq N,\ f(n) < g(n)$

i.e. f is eventually dominated by g

Comments ① Related to various other questions about the Turing degrees

E.g. Martin's Conjecture, a question of Day & Marks, etc

② There is a Borel function $F: 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $x' \leq_T y \Rightarrow F(x) <^* F(y)$

Answer to motivating question No.

Proof outline: ① Such an F exists $\Rightarrow$ $<_T$ is hyperfinite ??

② $<_T$ is not hyperfinite

Goal of this talk Introduce hyperfiniteness for Borel partial orders

② Hyperfiniteness

Def A partial order $(X, <)$ is:

locally finite if $\forall x, \{y \in X \mid y < x\}$ is finite

locally countable if $\forall x, \{y \in X \mid y < x\}$ is countable

Def A Borel partial order $(X, <)$ is hyperfinite if it is a countable increasing union of locally finite Borel partial orders

i.e. there are locally finite Borel partial orders $\{<_n\}_{n \in \mathbb{N}}$ on X such that

① $x <_n y \Rightarrow x <_{n+1} y$ — increasing

② $x < y \Rightarrow \exists n, x <_n y$ — $< \; = \bigcup_n <_n$

Comments ① Hyperfinite $\Rightarrow$ locally countable

② This definition works equally well for quasi-orders (generalizing hyperfinite BERs)

So do many (but not all) other things in this talk

Def A Borel partial order $(X, <)$ is hyperfinite if it is a countable increasing union of locally finite Borel partial orders

Example $<$ on $2^{\mathbb{N}}$ defined by $\quad 10111\ldots < 01111\ldots < 11111\ldots$

$$x < y \iff x =^* y \text{ and } x \neq y \text{ and for the largest } k \text{ s.t. } x(k) \neq y(k),\ x(k) < y(k)$$

eventually equal

Define $<_n$ by:

$$x <_n y \iff x < y \text{ and } \forall k \geq n,\ x(k) = y(k)$$

Comment If $<$ is generated by an $\mathbb{N}$-action then it is hyperfinite

$x \leq y \iff \exists n \in \mathbb{N},\ x = n \cdot y$

Question What about $\mathbb{N}^k$?

Question Generic hyperfiniteness?

③ Hyperwellfoundedness

<u>Thm</u> Suppose $(X,<)$ is a locally countable Borel partial order. Then $<$ is hyperfinite if and only if there is a Borel function $F: X \to \mathbb{N}^{\mathbb{N}}$ such that $x < y \Rightarrow F(x) <^* F(y)$

i.e. a Borel homomorphism from $<$ to $<^*$

Proof outline:

Hyperfinite $\Longrightarrow$ Borel hom. to $<^*$ $\Longrightarrow$ hyperwellfounded ??

<u>Comment</u> Very roughly:

hyperwellfounded $\approx$ hypersmooth

<u>Def</u> A Borel partial order $(X,<)$ is hyperwellfounded if it is a countable increasing union of well-founded Borel partial orders

Def A Borel partial order $(X, <)$ is hyperwellfounded if it is a countable increasing union of well-founded Borel partial orders

Example $<^*$ on $\mathbb{N}^{\mathbb{N}}$

For each n, define $<_n$ on $\mathbb{N}^{\mathbb{N}}$ by

$$f <_n g \iff \forall k \geq n,\ f(k) < g(k)$$

Well-foundedness:

$$f_0 >_n f_1 >_n f_2 >_n \dots \Rightarrow f_0(n) > f_1(n) > f_2(n) > \dots$$

Comments ① $<^*$ is "hyperheight $\leq \omega$"

② $<^*$ is not hyperfinite

Because it's not locally countable

3.1 Hyperfinite $\Rightarrow$ Hyperwellfounded

Assume: $(X, <)$ is a locally ctbl Borel partial order

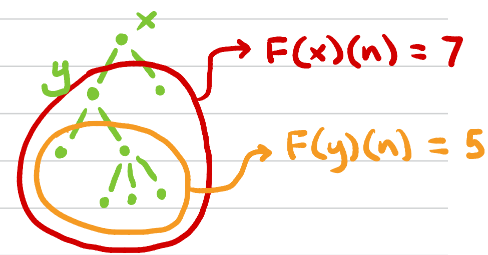
Prop If $<$ is hyperfinite then it has a Borel hom. to $<^*$

pf Let $\{<_n\}_n$ witness hyperfiniteness

Define $F : X \to \mathbb{N}^{\mathbb{N}}$

$F(x)(n) = |\{y \in X \mid y <_n x\}|$

Borel by Lusin-Novikov



Prop If $<$ has a Borel hom to $<^*$, then it is hyperwellfounded

pf Given $F : X \to \mathbb{N}^{\mathbb{N}}$, define

basically the proof that $<^*$ is hyperwellfdd

$$x <_n y \iff x < y \text{ and } \forall k \geq n,\ F(x)(k) < F(y)(k)$$

(3.2) Hyperwellfounded ⇒ Hyperfinite

Assume: $(X, <)$ is a locally ctbl Borel partial order

Prop If $<$ is hyperwellfounded then it is hyperfinite

Borel enumeration of predecessors

pf Fix $\{<_n\}_n$ witnessing hyperwellfoundedness

$\{g_n : X \to X\}_n$ Borel functions s.t.

$\forall x \ \{y \mid y < x\} = \{g_n(x) \mid n \in \mathbb{N}\}$

For each n, define

$$x \to_n y \iff x <_n y \text{ and } \exists k \leq n \ g_k(y) = x$$

and let $<_n'$ be the transitive closure of $\to_n$

Observation $\bigcup_n <_n' = <$

Claim $<_n'$ is locally finite

$\{<_n\}_n$ witness hyperwellfoundedness

$\{g_n : X \to X\}_n$ enumerate predecessors

$x \to_n y \iff x <_n y$ and $\exists k \le n \;\; g_k(y) = x$

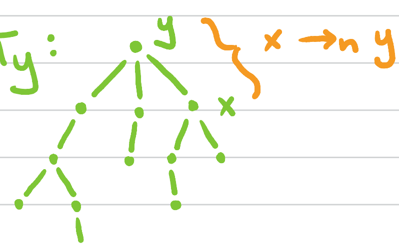
$<_n' =$ transitive closure of $\to_n$

<u>Claim</u> $<_n'$ is locally finite

pf Fix $y \in X$. WTS $\{x \mid x <_n' y\}$ is finite

Key point: $\{x \mid x <_n' y\}$ can be thought of as a finitely branching tree

T_y:



For each y, T_y is $\le n$-branching

T_y is infinite $\Rightarrow$ T_y ill-founded

$\Rightarrow$ $<_n$ ill-founded

3.3 Non-locally countable partial orders

<u>Thm</u> Suppose $(X, <)$ is a locally countable ? Borel partial order. Then $<$ is hyperfinite if and only if there is a Borel homomorphism from $<$ to $<^*$

Proof outline:

Hyperfinite $\Longrightarrow$ Borel hom. to $<^*$ $\Longrightarrow$ hyperwellfounded

implies loc. ctbl

Does not require local countability

What about the other (implicit) implication?

<u>Question</u> Does every not necessarily loc. ctbl hyperwellfounded Borel partial order have a Borel homomorphism to $<^*$?

My guess: Probably not

Hyperwellfounded should not imply "hyperheight $\leq \omega$"

④ Proving non-hyperfiniteness

Thm Suppose $(X,<)$ is a Borel partial order, μ is a Borel probability measure on X and $F_0, F_1 : X \to X$ are ?? μ-independent functions such that for μ-almost every x, $F_0(x) < x$ and $F_1(x) < x$. Then $<$ is not hyperwellfounded and hence not hyperfinite

Def Measurable functions $F, G : X \to X$ are μ-independent if for all measurable sets $A, B \subseteq X$

$$\mu(F^{-1}(A) \cap G^{-1}(B)) = \mu(A)\mu(B)$$

F, G μ-independent $\Leftrightarrow$ F, G μ-measure preserving and independent as random variables ??

Example $F, G : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ take left & right halves

$x = x_0 x_1 x_2 x_3 \ldots$

$$\left.\begin{array}{l} F(x) = x_0 x_2 x_4 \ldots \\ G(x) = x_1 x_3 x_5 \ldots \end{array}\right\} \text{independent for Lebesgue measure}$$

4.1 μ-independence

Assume: μ is a Borel probability measure on X

Def Measurable functions $F, G : X \to X$ are μ-independent if for all measurable sets $A, B \subseteq X$

$$\mu(F^{-1}(A) \cap G^{-1}(B)) = \mu(A)\mu(B)$$

Prop F, G μ-independent $\Rightarrow$ μ-measure preserving

pf $\mu(F^{-1}(A)) = \mu(F^{-1}(A) \cap G^{-1}(X)) \underset{\mu\text{-independence}}{=} \mu(A)\mu(X) \underset{\text{probability measure}}{=} \mu(A)$

Prop F, G μ-independent $\Rightarrow$ For any measurable $A, B \subseteq X$,

$$\mu(F^{-1}(A) \cup G^{-1}(B)) = \mu(A) + \mu(B) - \mu(A)\mu(B)$$

pf
$$\begin{aligned}\mu(F^{-1}(A) \cup G^{-1}(B)) &= \mu(F^{-1}(A)) + \mu(G^{-1}(B)) - \mu(F^{-1}(A) \cap G^{-1}(B)) \\ &= \mu(A) + \mu(B) - \mu(A)\mu(B)\end{aligned}$$

4.2 Proof of non-hyperfiniteness

Thm Suppose $(X, <)$ is a Borel partial order, μ is a Borel probability measure on X and $F_0, F_1 : X \to X$ are μ-independent functions such that for μ-almost every x, $F_0(x) < x$ and $F_1(x) < x$. Then $<$ is not hyperwellfounded

pf Suppose for contradiction that $<$ is hyperwellfdd
Let $\{<_n\}_n$ witness hyperwellfddness

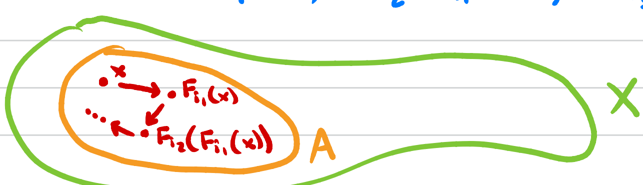
Pick n large enough that $\mu(A) \geq 3/4$, where

$$A = \{x \in X \mid F_0(x) <_n x \text{ and } F_1(x) <_n x\}$$

Goal: Find $x \in A$ and $i_1, i_2, i_3, \ldots \in \{0,1\}$ such that
$F_{i_1}(x), (F_{i_2} \circ F_{i_1})(x), (F_{i_3} \circ F_{i_2} \circ F_{i_1})(x), \ldots \in A$

$\Downarrow$

$x >_n F_{i_1}(x) >_n F_{i_2}(F_{i_1}(x)) >_n \ldots$



$(X,<)$: Borel partial order

μ: Borel probability measure on X

F_0, F_1: μ-independent s.t. for μ-a.e. x, $F_0(x), F_1(x) < x$

$\{<_n\}_n$: witness hyperwellfddness of X

A: $\{x \mid F_0(x) <_n x \text{ and } F_1(x) <_n x\}$, $\mu(A) \geq 3/4$

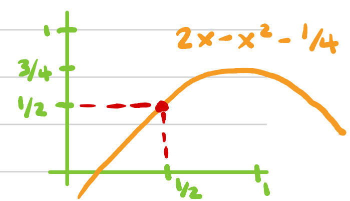
Goal: Find $x \in A$, $i_1, i_2, i_3, \ldots \in \{0,1\}$ s.t. $F_{i_1}(x), F_{i_2}(F_{i_1}(x)), \ldots \in A$

For each $k \in \mathbb{N}$, define

$$A_k = \{x \in A \mid \exists i_1, \ldots, i_k \,(F_{i_1}(x), \ldots, (F_{i_k} \circ \ldots \circ F_{i_1})(x) \in A)\}$$

<u>Claim</u> For all k, $\mu(A_k) > 1/2$

pf By induction. $A_0 = A \Rightarrow \mu(A_0) \geq 3/4 > 1/2$



$$A_{k+1} = A \cap (F_0^{-1}(A_k) \cup F_1^{-1}(A_k))$$

$$\Rightarrow \mu(A_{k+1}) \geq \mu(F_0^{-1}(A_k) \cup F_1^{-1}(A_k)) - 1/4$$

$$= \mu(A_k) + \mu(A_k) - \mu(A_k)\mu(A_k) - 1/4$$

By μ-independence

$$> 1/2$$

$(X,<)$: Borel partial order

μ: Borel probability measure on X

F_0, F_1: μ-independent s.t. for μ-a.e. x, $F_0(x), F_1(x) < x$

$\{<_n\}_n$: witness hyperwellfddness of X

A: $\{x \mid F_0(x) <_n x \text{ and } F_1(x) <_n x\}$, $\mu(A) \geq 3/4$

Goal: Find $x \in A$, $i_1, i_2, i_3, \ldots \in \{0,1\}$ s.t. $F_{i_1}(x), F_{i_2}(F_{i_1}(x)), \ldots \in A$

For each $k \in \mathbb{N}$, define

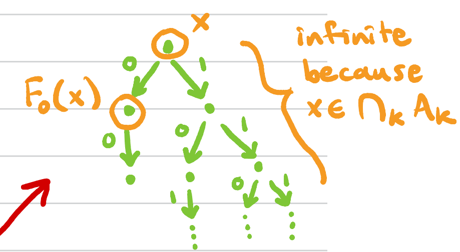
$$A_k = \{x \in A \mid \exists i_1, \ldots, i_k \, (F_{i_1}(x), \ldots, (F_{i_k} \circ \ldots \circ F_{i_1})(x) \in A)\}$$

<u>Claim</u> For all k, $\mu(A_k) > 1/2$

Claim $\Rightarrow$ $\mu(\bigcap_k A_k) \geq 1/2 \Rightarrow \bigcap_k A_k \neq \emptyset$

Note: $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$

Pick $x \in \bigcap_k A_k$



König's lemma $\Rightarrow$ $\exists i_1, i_2, i_3, \ldots$ s.t. $F_{i_1}(x), F_{i_2}(F_{i_1}(x)), \ldots \in A$

(4.3) Monoid actions

Thm Suppose $(X,<)$ is a Borel partial order, μ is a Borel probability measure on X and $F_0, F_1 : X \to X$ are μ-independent functions such that for μ-almost every x, $F_0(x) < x$ and $F_1(x) < x$. Then $<$ is not hyperwellfounded

On a set of measure 1, F_0, F_1 generate a free, measure-preserving action of the free monoid on 2 generators and the associated partial order is a suborder of $<$

Question Is independence necessary?

Suppose $F_0, F_1 : X \to X$ generate a free, μ-measure-preserving action of the free monoid on 2 generators. Can the associated partial order be hyperfinite?

Answer (Forte Shinko) Yes.

Question What is the appropriate notion of independence for actions of non-free monoids?

⑤ Turing reducibility

Motivating question: Is there a Borel function $F: 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $x, y \in 2^{\mathbb{N}}$, $x <_T y \Rightarrow F(x) <^* F(y)$?

Thm No such function exists

pf Equivalent: $<_T$ is not hyperwellfounded

Let $F, G: 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ take left & right halves

i.e. if $x = x_0 x_1 x_2 x_3 \ldots$ then $F(x) = x_0 x_2 x_4 \ldots$ $G(x) = x_1 x_3 x_5 \ldots$

Well-known fact: For almost every x, $F(x), G(x) <_T x$

Mentioned previously: F, G independent for Lebesgue measure

$\Rightarrow <_T$ is not hyperwellfounded

Thm There is no Borel function $F : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $x, y \in 2^{\mathbb{N}}$, $x <_T y \Rightarrow F(x) <^* F(y)$

Prop There <u>is</u> a Borel function $F : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $x, y \in 2^{\mathbb{N}}$, $x' \leq_T y \Rightarrow F(x) <^* F(y)$

Cor The partial order $x' \leq_T y$ is hyperfinite

pf (of Prop) For each $x \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, define

$$F(x)(n) = \max \{ \varphi_k^x(n) \mid k \leq n \text{ and } \varphi_k^x(n)\downarrow \} + 1$$

i.e. $F(x)$ eventually dominates each x-computable function

Suppose $x' \leq_T y$. $F(x) \leq_T x' \Rightarrow F(x) \leq_T y \Rightarrow F(x) <^* F(y)$

Question Is $\ll$ hyperfinite?

Recall $x \ll y$ means y is of PA degree over x