The rank problem for complete, separable metric spaces

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Scott's analysis of a metric space

In a research note from 2012, Fokina, Friedman, Koerwien and Nies consider metric spaces as classical model-theoretic structures as follows:

- $\mathcal{L}$ is a language where for every $q \in \mathbb{Q}_+$ there are binary relation symbols $R_{<q}$ and $R_{>q}$.
- Given a metric space (X, d_X), we turn it into an $\mathcal{L}$-structure $\mathfrak{X}$ by defining

$$R^{\mathfrak{X}}_{<q} = \{(x, y) \in X^2 \mid d(x, y) < q\}$$
$$R^{\mathfrak{X}}_{>q} = \{(x, y) \in X^2 \mid d(x, y) > q\}$$

Fokina, Friedman, Koerwien and Nies (FFKN) then consider the **classical Scott analysis** of such structures $\mathfrak{X}$ (details on next slide), and ask:

Question (Fokina-Friedman-Koerwien-Nies)

Is the Scott rank of a **complete, separable** metric space countable?

Scott analysis of $\mathfrak{X}$ || **Note:** (X, d_X) fixed metric space.

Notation: $X^{<\infty}$ =the set of finite sequences from X.

The Scott analysis of $\mathfrak{X}$ will give a transfinite sequence

$$\simeq_0^{\mathsf{FFKN}} \supseteq \simeq_1^{\mathsf{FFKN}} \supseteq \dots \supseteq \simeq_\alpha^{\mathsf{FFKN}} \supseteq \dots$$

of equivalence relations on $X^{<\infty}$, and the $\simeq_\alpha^{\mathsf{FFKN}}$ approximate from above the equivalence relation

$$\bar{a} \simeq_{isomet} \bar{b} \text{ iff there is an autoisometry of } (X, d_X) \text{ mapping } \bar{a} \text{ to } \bar{b}.$$

Definition

For $\bar{a} = (a_0, \dots, a_{n-1}) \in X^{<\infty}$, the $n \times n$ **distance matrix** $\underline{D}^{d_X, \bar{a}}$ of $\bar{a}$ is

$$\underline{D}_{i,j}^{d_X, \bar{a}} = d_X(a_i, a_j).$$

Definition of $\simeq_0^{\mathsf{FFKN}}$

$\bar{a} \simeq_0^{\mathsf{FFKN}} \bar{b}$ iff $\underline{D}^{d_X, \bar{a}} = \underline{D}^{d_X, \bar{b}}$ iff $\bar{a}$ and $\bar{b}$ have the same *quantifier-free* type.

Definition of $\simeq_\alpha^{\mathsf{FFKN}}$ || **Note:** (X, d_X) fixed metric space.

Recursive definition of $\simeq_\alpha^{\mathsf{FFKN}}$

By recursion on $\alpha \in \mathbb{ON}$, define:

1. $\bar{a} \simeq_0^{\mathsf{FFKN}} \bar{b}$ is already defined to mean $\underline{D}^{d_X,\bar{a}} = D^{d_X,\bar{b}}$.
2. $\bar{a} \simeq_{\alpha+1}^{\mathsf{FFKN}} \bar{b}$ iff
   - $\forall x \in X\ \exists y \in X\ \ \bar{a}^\frown x \simeq_\alpha^{\mathsf{FFKN}} \bar{b}^\frown y$ **and**
   - $\forall y \in X\ \exists x \in X\ \ \bar{b}^\frown y \simeq_\alpha^{\mathsf{FFKN}} \bar{a}^\frown x$.
3. $\bar{a} \simeq_\lambda^{\mathsf{FFKN}} \bar{b}$ iff $\forall \alpha < \lambda\ \bar{a} \simeq_\alpha^{\mathsf{FFKN}} \bar{b}$ when λ is a limit ordinal.

Remark: Suppose (X, d_X) is **separable**. If there is $\alpha^* \in \mathbb{ON}$ such that

$$\bar{a} \simeq_{\alpha^*}^{\mathsf{FFKN}} \bar{b} \implies \forall n > 0\ \forall \bar{u}, \bar{v} \in X^n\ \forall \beta\ \ \bar{a}^\frown \bar{u} \simeq_\beta^{\mathsf{FFKN}} \bar{b}^\frown \bar{v},$$

then a standard argument shows that there is an autoisometry of (X, d_X) which maps $\bar{a}$ to $\bar{b}$.

This motivates the definition of rank on the next slide.

Definition of $\mathrm{rk}^{\mathrm{FFKN}}$

1. For $\bar{a} \in X^{<\infty}$ of length > 0, define:

   $\mathrm{rk}^{\mathrm{FFKN}}(\bar{a})$ = the **least** α such that

   $$\forall \bar{b} \ \ \bar{a} \simeq_{\alpha}^{\mathrm{FFKN}} \bar{b} \ \text{ implies } \ \forall \beta \ \ \bar{a} \simeq_{\beta}^{\mathrm{FFKN}} \bar{b}.$$

2. $\mathrm{rk}(X, d_X) = \sup\{\mathrm{rk}^{\mathrm{FFKN}}(\bar{a}) \mid \bar{a} \in X^{<\infty} \wedge \mathrm{lh}(\bar{a}) > 0\}$.

Remark: A cardinality argument shows that $\mathrm{rk}^{\mathrm{FFKN}}(\bar{a})$ exists.

Question (Fokina-Friedman-Koerwien-Nies)

Is $\mathrm{rk}^{\mathrm{FFKN}}(X, d_X)$ **countable** when (X, d_X) is **complete and separable**?

- M. Doucha: $\mathrm{rk}^{\mathrm{FFKN}}(X, d_X) \leq \omega_1$ for (X, d_X) complete and separable.
- W. Chan: "Yes" if we impose some rigidity assumptions on (X, d_X).

Theorem (T., 2026)

There is a complete, separable metric space of rank ω_1.

Digraphs, Rado digraphs, coding of digraphs

Digraphs and Rado digraphs

The metric space $(\mathrm{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$ is obtained from the set of self-embeddings of the **Rado digraph** (see below) with some additional structure added.

It is easier to work with digraphs (i.e., directed graphs), rather than graphs, because ordinals (with their usual ordering) are digraphs.

Definition

1. A **digraph** is a pair $(|Q|, Q)$ consisting of a set $|Q|$, the **domain** of the digraph, and an **irreflexive** relation $Q \subseteq |Q|^2$. For $x \in |Q|$, we let
$$Q_x = \{y \in |Q| \mid (x, y) \in Q\} \text{ and } Q^x = \{y \in |Q| \mid (y, x) \in Q\}$$
2. A digraph R (where $|R|$ is allowed to have **any** infinite cardinality) will be called a **Rado digraph** if whenever we're given *finite* sets $A, B \subseteq C \subseteq |R|$ there is $x \in |R| \setminus C$ such that
$$R_x \cap C = A \text{ and } R^x \cap C = B.$$

Up to isomorphism, there is a unique **countable** Rado digraph.

For this talk, **let $\mathfrak{R}$ be a fixed countable Rado digraph with** $|\mathfrak{R}| = \omega$.

Rado extension

It is easy to prove that any digraph can be embedded into a Rado digraph.

It'll be useful that this can done in the following "nice way":

Definition

Let Q, R be digraphs with $|Q| \subseteq |R|$. We say that R is a **Rado extension** of Q, written $Q \sqsubset_r R$, if $Q = R \cap |Q|^2$ and the following two hold:

1. For all finite $A, B \subseteq C \subseteq |R|$ there is $x \in |R| \setminus (|Q| \cup C)$ such that
$$R_x \cap (C \cup |Q|) = A \text{ and } R^x \cap (C \cup |Q|) = B.$$
2. For all $x \in |R| \setminus |Q|$, the sets $R_x \cap |Q|$ and $R^x \cap |Q|$ are finite.

Proposition (Standard fact)

If $Q \sqsubset_r \mathfrak{R}$ and $S \sqsubset_r \mathfrak{R}$ and $\psi : Q \leftrightarrow S$ is an isomorphism, then ψ extends to an automorphism of $\mathfrak{R}$.

Warning: Not generally true if $\mathfrak{R}$ is replaced by uncountable Rado digraph.

'Orthogonal' and coding with embeddings (Melleray-style)

Definition: Let Q and S be digraphs. Given $D \subseteq |Q|$, define

$$D^{\perp Q} = \{x \in |Q| \mid x \notin D \wedge Q_x \cap D = \emptyset \wedge Q^x \cap D = \emptyset\}.$$

If Q is understood, we write $D^\perp$. **So:** "$D^\perp = |Q| \setminus (\partial D \cup D)$".

Definition (Recall: $\mathfrak{R}$ is our fixed Rado digraph on ω)

Let Q be a digraph, R a Rado digraph. We say the digraph embedding $F : \mathfrak{R} \hookrightarrow R$ **codes** Q if

1. Q is isomorphic to $R \restriction \operatorname{ran}(F)^{\perp R}$.
2. $R \restriction (\operatorname{ran}(F) \cup (\operatorname{ran} F)^{\perp R}) \sqsubset_r R$.

Proposition

1. *Every countable digraph Q is coded by some $F_Q : \mathfrak{R} \hookrightarrow \mathfrak{R}$.*
2. *If $F_Q, F_S : \mathfrak{R} \hookrightarrow \mathfrak{R}$ code Q and S, respectively, then:*

$$Q \text{ is isomorphic to } S \quad \text{iff} \quad \exists \sigma \in \operatorname{Aut}(\mathfrak{R})\ \sigma \circ F_Q = F_S.$$

Definition of the metric space $(\mathsf{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$

The underlying set of $(\mathrm{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$

Definition

1. For digraphs Q and S, let
$$\mathrm{Emb}(Q,S) = \{F : |Q| \hookrightarrow |S| \mid F \text{ embeds } Q \text{ into } S\}.$$
2. We let $\mathcal{F}$ denote **the set of all digraphs whose domains are non-empty finite ordinals** (i.e., are in $\omega \setminus \{0\}$)
3. For S a digraph, we let
$$\mathrm{Emb}(\mathcal{F},S) = \{(P,f) \mid P \in \mathcal{F} \text{ and } f \text{ embeds } P \text{ into } S\}.$$

The set $\mathrm{E}_{\mathfrak{R}}$ and the idea

The underlying set of the metric space $(\mathrm{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$ we will define is
$$\mathrm{E}_{\mathfrak{R}} = \mathrm{Emb}(\mathfrak{R},\mathfrak{R}) \cup \mathrm{Emb}(\mathcal{F},\mathfrak{R}).$$
Idea: • Elements of $\mathrm{Emb}(\mathfrak{R},\mathfrak{R})$ **code arbitrary countable digraphs** as we have seen, in particular **countable ordinals**.
• The distances in and from $\mathrm{Emb}(\mathcal{F},\mathfrak{R})$ to $F_Q \in \mathrm{Emb}(\mathfrak{R},\mathfrak{R})$ coding Q will **recover the classical Scott analysis** of the countable digraph Q.

Defining $d_{\mathfrak{R}}$ (and more generally d_R)

Over the next several slides, we'll define the desired $d_{\mathfrak{R}}$

1. First on $\mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$.
2. Then on $\mathrm{Emb}(\mathcal{F}, \mathfrak{R})$.
3. Then finally we define the distancs between points in $\mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$ and points in $\mathrm{Emb}(\mathcal{F}, \mathfrak{R})$.

The first item on the list will be easy (next slide), but for the two next items on the list will use a function δ that we will define when we get there.

For reasons that has to do with the proof that $(\mathrm{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$ has rank ω_1, it will be important not just to define the metric $d_{\mathfrak{R}}$ on

$$\mathrm{E}_{\mathfrak{R}} = \mathrm{Emb}(\mathfrak{R}, \mathfrak{R}) \cup \mathrm{Emb}(\mathcal{F}, \mathfrak{R}),$$

but more generally a metric d_R on

$$\mathrm{E}_R = \mathrm{Emb}(\mathfrak{R}, R) \cup \mathrm{Emb}(\mathcal{F}, R)$$

where R is an arbitrary Rado digraph **of any size**.

The metric $\mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$

In general, for a digraph R, we equip $\mathrm{Emb}(\mathfrak{R}; R)$ with the metric

$$d_R^*(F, G) = \begin{cases} 0 & \text{if } F = G; \\ 2^{-\min\{k+1 \in \omega \mid F(k) \neq G(k)\}} & \text{if } F \neq G. \end{cases}$$

Note that d_R^* induces the pointwise convergence topology, and that when R is countable then $(\mathrm{Emb}(\mathfrak{R}; R), d_R^*)$ is a complete separable metric space.

Definition

On $\mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$ the metric $d_{\mathfrak{R}}$ is defined to agree with $d_{\mathfrak{R}}^*$.

Next we need to define $d_{\mathfrak{R}}$-distances on $\mathrm{Emb}(\mathcal{F}, \mathfrak{R})$ and between $\mathrm{Emb}(\mathcal{F}, \mathfrak{R})$ and $\mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$.

Note: $\mathrm{ran}(d_R^*) = \{\frac{1}{2^n} \mid n \in \omega \wedge n > 0\} \subseteq [0, \frac{1}{2}]$.

Immediate extension and the function $\delta : \mathcal{F} \cup \sqsubset \to (1, \frac{3}{2})$

Recall that $\mathcal{F}$ is the set of all digraphs whose domains are non-0 finite ordinals, and that

$$\mathrm{Emb}(\mathcal{F}, R) = \{(P, f) \mid P \in \mathcal{F} \text{ and } f \text{ embeds } P \text{ into } R\}.$$

Note: We use f to refer to $(P, f) \in \mathrm{Emb}(\mathcal{F}, R)$, and $\mathfrak{dom}(f)$ refers to P.

Definition

The **immediate extension** relation $\sqsubset$ in $\mathcal{F}$ is defined as

$$\sqsubset = \{(P, Q) \in \mathcal{F}^2 \mid |Q| = |P| + 1 \wedge |P|^2 \cap Q = P\}.$$

The function δ

In order to define metrics on $\mathrm{Emb}(\mathcal{F}; R)$ and E_R we fix an **injection** $\delta : \mathcal{F} \cup \sqsubset \to (1, \frac{3}{2})$ such that:

1. If $x, y \in \mathrm{ran}(\delta)$ and $x - y$ is a dyadic rational, then $x = y$.
2. δ takes rational values, that is, $\mathrm{ran}(\delta) \subseteq \mathbb{Q} \cap (1, \frac{3}{2})$.
3. δ is recursive (or at least $\delta \in L$).

The metric $\mathrm{Emb}(\mathcal{F}, \mathfrak{R})$ and $\mathrm{Emb}(\mathcal{F}, R)$

Definition

We define $d_R^{**} : \mathrm{Emb}(\mathcal{F}; R)^2 \to \mathbb{R}_{\geq 0}$ to be the symmetric function such that

$$d_R^{**}(f, g) = \begin{cases} 0 & \text{if } f = g; \\ \delta(\mathfrak{dom}(f), \mathfrak{dom}(g)) & \text{if } \mathfrak{dom}(f) \sqsubset \mathfrak{dom}(g) \wedge f \subset g; \\ 1 & \text{otherwise.} \end{cases}$$

Proposition

1. *d_R^{**} is a metric, which induces the discrete topology on* $\mathrm{Emb}(\mathcal{F}; R)$.
2. *In particular, if R is a countable digraph, then* $\mathrm{Emb}(\mathcal{F}; R)$ *is countable, and* $(\mathrm{Emb}(\mathcal{F}; R), d_R^{**})$ *is separable and complete.*

Definition

On $\mathrm{Emb}(\mathcal{F}, \mathfrak{R})$, we define $d_{\mathfrak{R}}$ to agree with $d_{\mathfrak{R}}^{**}$.

Distances between points in $\mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$ and $\mathrm{Emb}(\mathcal{F}, \mathfrak{R})$

For $(P, f) \in \mathrm{Emb}(\mathcal{F}, R)$, let

$$C_{\mathrm{ran}(f)} = \{F \in \mathrm{Emb}(\mathfrak{R}, R) \mid \mathrm{ran}(f) \subseteq \mathrm{ran}(F)^{\perp}\}$$

This set is closed in the pointwise convergence topology (the complement is clearly open).

Definition

We define $d_R : \mathrm{E}_R^2 \to [0, 2]$ to be the symmetric function extending d_R^{**} and d_R^{*} such that

$$d_R(f, F) = d_R(F, f) = d_R^*(F, C_{\mathrm{ran}(f)}) + \delta(\mathfrak{dom}(f)),$$

where

$$d_R^*(F, C_{\mathrm{ran}(f)}) = \inf\{d_R^*(F, G) \mid G \in C_{\mathrm{ran}(f)}\}.$$

In particular, $d_{\mathfrak{R}}$ on $\mathrm{E}_{\mathfrak{R}}$ is now defined.

Basic decoding with d_R

Proposition

(A) *d_R is a complete metric on E_R, and if $F \in Emb(\mathfrak{R}, R)$ and $f \in \mathrm{Emb}(\mathcal{F}, R)$ then*

$$d_R(F, f) \geq 1.$$

(B) *$d_R(x, y) = \delta(\mathfrak{dom}(f))$ if and only if one of the following hold:*

1. $x = f$, $y \in \mathrm{Emb}(\mathfrak{R}; R)$, *and* $\mathrm{ran}(f) \subseteq y^{\perp}$.
2. $y = f$, $x \in \mathrm{Emb}(\mathfrak{R}; R)$, *and* $\mathrm{ran}(f) \subseteq x^{\perp}$.

(C) *If R is countable, then E_R is a separable complete metric space.*

Remark

(B) above shows that the distances between $F \in \mathrm{Emb}(\mathfrak{R}, R)$ and $f \in \mathrm{Emb}(\mathcal{F}, R)$ can detect if $\mathrm{ran}(f) \subseteq \mathrm{ran}(F)^{\perp}$. From this one can verify when $F_Q, F_S \in \mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$ code the countable digraphs Q and S, then

$$F_Q \simeq_{\alpha}^{\mathrm{FFKN}} F_S \text{ implies that } Q \simeq_{\alpha}^{Scott} S.$$

A very few words about proving that

$$\mathsf{rk}^{\mathsf{FFKN}}(\mathsf{E}_{\mathfrak{R}}, d_{\mathfrak{R}}) = \omega_1$$

The plan now

- Since the ordering of an ordinal makes the ordinal a digraph, we have that for each countable ordinal β there is

$$F_\beta \in \mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$$

which codes β (up to isomorphism).

- The plan is to prove that in the metric space $(\mathrm{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$ we have

$$\sup\{\mathrm{rk}^{\mathrm{FFKN}}(F_\beta) \mid \beta < \omega_1\} = \omega_1,$$

thereby proving that $\mathrm{rk}^{\mathrm{FFKN}}(\mathrm{E}_{\mathfrak{R}}, d_{\mathfrak{R}}) = \omega_1$.

The challenge is that the FFKN-Scott analysis of F_β in the metric space $(\mathrm{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$ could pick up far more information about β than the classical Scott analysis does.

The schematic nature of Rado extensions

The key thing to notice is that while elements of $\mathrm{E}_{\mathfrak{R}}$ can only code **countable** ordinals, we can describe schematically embeddings $F_\beta : \mathfrak{R} \to R_\beta$ where

- R_β is a Rado digraph (uncountable when β is uncountable);
- $F_\beta : \mathfrak{R} \to R_\beta$ codes β;
- $\beta \mapsto F_\beta$ and $\beta \mapsto R_\beta$ are class functions definable without parameters (indeed are in L).

Connection to Stern's absoluteness

Notice now that in a forcing extension where β becomes countable, R_β will become isomorphic to $\mathfrak{R}$, and so F_β will essentially become an element of $\mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$.

So we have described in L objects F_β that can be considered "virtual" elements of $\mathrm{Emb}(\mathfrak{R}, \mathfrak{R})$, much in the same way as J. Stern in his classic paper *'On Lusin's restricted continuum problem'* described "virtual" Borel sets (that only become Borel sets after collapsing a cardinal).

Absoluteness to a a countable transitive model

Unfortunately, the equivalence relations $\simeq_\alpha^{\mathsf{FFKN}}$ have high complexity in the projective hierarchy, so where Stern has the advantage that Borel sets have good absoluteness properties, we don't. But this can be sidestepped:

Lemma (For $\simeq_\alpha^{\mathsf{FFKN}}$ on the space $(\mathsf{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$)

For any countable $\alpha^* < \omega_1$, there is a countable transitive model M (of a large fragment of ZFC) with $\alpha^* \in M$, such that for all $\alpha, \beta, \gamma \in \mathbb{ON} \cap M$:

1. $F_\beta, F_\gamma \in M$ *and* $R_\beta, R_\gamma \in M$.
2. $(F_\beta \simeq_\alpha^{\mathsf{FFKN}} F_\gamma)^M \iff F_\beta \simeq_\alpha^{\mathsf{FFKN}} F_\gamma$

Using this lemma, we'll sketch the proof that $\mathrm{rk}^{\mathsf{FFKN}}(\mathsf{E}_{\mathfrak{R}}, d_{\mathfrak{R}}) = \omega_1$ *on the next slide.*

A cardinality argument

We repeat the Lemma:

Lemma (For $\simeq_\alpha^{\mathsf{FFKN}}$ on the space $(\mathsf{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$)

For any countable $\alpha^* < \omega_1$, there is a countable transitive model M (of a large fragment of ZFC) with $\alpha^* \in M$, such that for all $\beta, \gamma \in \mathbb{ON} \cap M$:

1. $F_\beta, F_\gamma \in M$ *and* $R_\beta, R_\gamma \in M$.
2. $(F_\beta \simeq_\alpha^{\mathsf{FFKN}} F_\gamma)^M \iff F_\beta \simeq_\alpha^{\mathsf{FFKN}} F_\gamma$

Proof that $(\mathsf{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$ has rank ω_1: Suppose $\mathrm{rk}^{\mathsf{FFKN}}(\mathsf{E}_{\mathfrak{R}}, d_{\mathfrak{R}}) = \alpha^* < \omega_1$. Let M be as in the Lemma. A cardinality argument in M shows that there are $\beta, \gamma \in \mathbb{ON} \cap M$ (so β, γ are countable) with $\beth_{\alpha^*}^M < \beta < \gamma$ such that

$$(F_\beta \simeq_{\alpha^*}^{\mathsf{FFKN}} F_\gamma)^M.$$

Then $F_\beta \simeq_{\alpha^*}^{\mathsf{FFKN}} F_\gamma$, so $F_\beta \sim_{isomet} F_\gamma$ since $\alpha^* = \mathrm{rk}^{\mathsf{FFKN}}(\mathsf{E}_{\mathfrak{R}}, d_{\mathfrak{R}})$. Now we obtain a contradiction as follows: $\beta \neq \gamma$, so for some $\xi \in \mathbb{ON}$ we have $\beta \not\simeq_\xi^{\mathsf{Scott}} \gamma$, but by the Remark on slide 17 we then have $F_\beta \not\simeq_\xi^{\mathsf{FFKN}} F_\gamma$, contradicting that $F_\beta \sim_{isomet} F_\gamma$. □

Thanks for listening!