Generalized descriptive set theory at singular cardinals of countable cofinality

Luca Motto Ros

Department of Mathematics "G. Peano"
University of Turin, Italy
luca.mottoros@unito.it
https://sites.google.com/site/lucamottoros/

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What is generalized descriptive set theory?

The goal of *classical* descriptive set theory is to study definable objects (sets, functions, equivalence relations, etc.) on Polish or standard Borel spaces, like the Baire space ${}^\omega\omega$ and the Cantor space ${}^\omega 2$. The fact that Polish spaces are ubiquitous in mathematics is one of the reasons behind the success of this area in the last few decades. However, the restriction to separable spaces seems a serious limitation.

Question

To what extent can the DST ideas and methods be adapted to the context of nonseparable spaces?

This natural problem led to various (sometimes incompatible) theories, each of which has been associated with the name "*generalized* descriptive set theory".

Approach 1: general topology

Given an uncountable cardinal λ, consider the λ-**Baire space** $B(\lambda) = {}^{\omega}\lambda$ and, if $\mathrm{cof}(\lambda) = \omega$, the λ-**Cantor space** $C(\lambda) = \prod_{i<\omega} \lambda_i$, where $\lambda_i \nearrow \lambda$; both are completely metrizable spaces of weight λ.

Stone, 1962

Study of (classical) Borel subsets of $B(\lambda)$ and $C(\lambda)$, and of the "λ-analytic sets" (= continuous images of $B(\lambda)$).

Problem: There is a serious mismatch between the two notions (in particular, there is no Lusin-Suslin theorem).

Hansell, 1972-73

Consider instead "hyper-Borel sets" and "λ-Suslin sets", defined through σ-discrete unions.

Problem: Although one now gets a Lusin-Suslin-like theorem, it was soon discovered that Hansell's λ-Suslin sets coincide with the classical analytic sets, a class much smaller than the one of λ-analytic sets.

Approach 2: set theory and model theory

Replace ω with any uncountable cardinal λ *satisfying* $\lambda^{<\lambda} = \lambda$ in all basic DST definitions.

Vaught, 1975; Mekler-Väänänen, 1993; many others since 2014

The **generalized Baire space** is the space ${}^{\lambda}\lambda$ whose topology is generated by the sets $\boldsymbol{N}_s = \{x \in {}^{\lambda}\lambda \mid x \supseteq s\}$, for $s \in {}^{<\lambda}\lambda$. Its subspace ${}^{\lambda}2$ is the **generalized Cantor space**. One consider e.g. λ^+-Borel sets instead of classical Borel sets, and so on.

There are beautiful connections with model theory, however:

- The assumption $\lambda^{<\lambda} = \lambda$ excludes singular cardinals, but unfortunately ${}^{\lambda}\lambda$ is metrizable if and only if $\mathrm{cof}(\lambda) = \omega < \lambda$.
- All basic results are either false (like the Lusin-Suslin theorem), or independent of ZFC.
- Heuristically: the larger is λ, the more stable is the resulting generalized DST. But the largest cardinal assumptions conceived so far (e.g. I0) imply that the cardinal at hand has countable cofinality!

Approach 3: very large cardinals

Given a limit cardinal λ, let $\mathsf{I0}(\lambda)$ be the assertion: There is a nontrivial elementary embedding $j\colon L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$.

Woodin, 2011

Equip $V_{\lambda+1}$ with the topology generated by the sets

$$O_{(a,\alpha)} = \{A \subseteq V_\lambda \mid A \cap V_\alpha = a\}, \qquad \alpha < \lambda,\ a \subseteq V_\alpha.$$

Woodin proved that there is a striking analogy between the structure under $\mathsf{I0}(\lambda)$ of the subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$, and the structure under AD of the subsets of, say, ${}^{\omega}2$ in $L(\mathbb{R})$. However:

- the analogy is not developed further, and does not include complexity hierarchies like the (appropriate generalizations of the) Borel or the projective ones;
- the space $V_{\lambda+1}$ is a bit exotic, and it is unclear whether the theory can be applied to any other meaningful space.

A unifying approach

Key observation

Under the appropriate assumptions on λ,
the three approaches concern the same spaces!

Let $\lambda > \omega$ be a limit cardinal with countable cofinality.

- $B(\lambda) \approx C(\lambda)$ (independently of the chosen cofinal sequence $\lambda_i \nearrow \lambda$);
- if $2^{<\lambda} = \lambda$ (equivalently: λ is strong limit), then ${}^{\lambda}2 \approx B(\lambda) \approx C(\lambda)$;
- if $\beth_\lambda = \lambda$ (equivalently: $|V_\lambda| = \lambda$), then $V_{\lambda+1} \approx {}^{\lambda}2 \approx B(\lambda) \approx C(\lambda)$.

Remarks

1. The cardinal assumptions above are "minimal": they ensure that the spaces at hand have weight λ.
2. The "right" generalized Baire space is ${}^{\mathrm{cof}(\lambda)}\lambda$, and not (as previously believed) the space ${}^{\lambda}\lambda$.
3. The λ-Cantor space $C(\lambda)$ and the generalized Cantor space ${}^{\lambda}2$ are not compact, and indeed they are not even λ-Lindelöf.

λ-Polish spaces

Definition

A topological space is λ-**Polish** if it is completely metrizable and has weight at most λ.

Examples:

- All the previously mentioned spaces: $B(\lambda)$, $C(\lambda)$, ${}^{\lambda}2$, and $V_{\lambda+1}$ (under the appropriate assumptions on λ).
- All discrete spaces of weight/size at most λ.
- All Banach spaces of weight at most λ.
- The Vietoris space $\mathcal{K}(X)$, for X λ-Polish.

One can check that λ-Polish spaces are closed under G_δ spaces, disjoint unions of size λ, and countable products.

Warning!

When generalizing classical definitions, we need to carefully choose when to replace ω with λ, or with its cofinality $\mathrm{cof}(\lambda) = \omega$.

Some "standard" results

Let X be a λ-Polish space.

1. There is a closed set $F \subseteq B(\lambda)$ and a continuous bijection $f \colon F \to X$.
2. There is a continuous open surjection $g \colon B(\lambda) \to X$.
3. There is a closed $F \subseteq B(\lambda)$ and a continuous surjection $h \colon F \to X$ such that $h^{-1}(K) \in \mathcal{K}(B(\lambda))$ for every $K \in \mathcal{K}(X)$.
4. If λ is ω-inaccessible ($= \forall \kappa < \lambda \, (\kappa^\omega < \lambda)$), then $B(\lambda)$ embeds in X *as a closed set* as soon as X is λ-**perfect**, that is, no point in X is λ-isolated ($=$ has an open neighborhood of size $< \lambda$).
5. If λ is ω-inaccessible, then X can be uniquely decomposed into its λ-**perfect kernel** P and an open set C of size at most λ. Therefore either $|X| \leq \lambda$, or $B(\lambda)$ embeds into X as a closed set. (This is the generalized Cantor-Bendixson theorem.)

Zero-dimensionality

We use Lebesgue covering dimension, instead of the inductive dimension.

Definition

We write $\dim(X) = 0$ if every open covering of X can be refined to a clopen partition.

If X is λ-Polish, then $\dim(X) = 0$ iff $X \approx F$ for some closed $F \subseteq B(\lambda)$.

Characterizations of $B(\lambda)$, and its relatives

$B(\lambda)$ is the unique (nonempty) λ-Polish space X such that:

- $\dim(X) = 0$;
- X is everywhere of weight λ.

The second item can be replaced by any of the following:

- every λ-Lindelöf subspace of X has empty interior;
- X is λ-perfect;
- X has weight λ and is h-homogeneous.

Generalized Borel sets

Following Approach 2, we consider the following notion.

Definition

A set $A \subseteq X$ is λ-**Borel** if it is in the smallest λ^+-algebra (equivalently: λ-algebra) $\lambda\text{-}\mathbf{Bor}$ generated by the open sets of X.

Clearly, $\lambda\text{-}\mathbf{Bor}$ can be stratified in the classes $\lambda\text{-}\mathbf{\Sigma}^0_\xi$, $\lambda\text{-}\mathbf{\Pi}^0_\xi$, and $\lambda\text{-}\mathbf{\Delta}^0_\xi$, for $1 \leq \xi < \lambda^+$. In most cases, such classes work as expected: in particular, the λ-Borel hierarchy does not collapse on λ-Polish spaces X with $|X| > \lambda$ (assuming $2^{<\lambda} = \lambda$).

Warning!

Some quirks may occur. For example, $\lambda\text{-}\mathbf{\Sigma}^0_\xi$ is closed under finite intersections, but not under intersections of size $< \lambda$.

Remark: Classical Borel sets (as considered e.g. by Stone) are contained in $\lambda\text{-}\mathbf{\Delta}^0_2$; indeed, when $|X| > \lambda = 2^{<\lambda}$ we have $\nu\text{-}\mathbf{Bor} \subsetneq \lambda\text{-}\mathbf{\Delta}^0_2$ for every $\nu < \lambda$.

Changes of topology and structural properties

Theorem

Assume $2^{<\lambda} = \lambda$. For every λ-Borel $B \subseteq X$ there is a closed $F \subseteq B(\lambda)$ and a continuous λ-Borel isomorphism $f \colon F \to X$ such that $f^{-1}(B)$ is clopen relatively to F.

Consequently, one can turn λ-Borel sets into clopen sets without altering the λ-Borel structure of the space.

Theorem

Assume $2^{<\lambda} = \lambda$ and $1 < \xi < \lambda^+$.

- λ-$\mathbf{\Sigma}^0_\xi$ has the ordinal λ-uniformization property and the (λ-generalized) reduction property, but not the separation property.
- λ-$\mathbf{\Pi}^0_\xi$ has the (λ-generalized) separation property, but not the reduction property.

The same is true for $\xi = 1$ if $\dim(X) = 0$.

λ-analytic sets

The following definition is equivalent to the one introduced by Stone in 1962 (Approach 1).

Definition

Let X be λ-Polish. A set $A \subseteq X$ is λ-**analytic**, or λ-$\mathbf{\Sigma}^1_1$, if it is a continuous image of some λ-Polish space Y.

A set $A \subseteq X$ is λ-**coanalytic**, or λ-$\mathbf{\Pi}^1_1$, if $X \setminus A \in \lambda$-$\mathbf{\Sigma}^1_1$;
it is λ-**bianalytic**, or λ-$\mathbf{\Delta}^1_1$, if $A \in \lambda$-$\mathbf{\Sigma}^1_1 \cap \lambda$-$\mathbf{\Pi}^1_1$.

A typical example of a (complete) λ-analytic set is the set IF_λ of (codes for) ill-founded trees $T \subseteq {}^{<\omega}\lambda$.

Warning!

This would fail if $\operatorname{cof}(\lambda) > \omega$, as in that case well-foundedness would be a clopen condition.

Generalized Lusin-Suslin theorem

Proposition 1

Assume $2^{<\lambda} = \lambda$. Then λ-$\mathbf{\Sigma}^1_1$ is closed under continuous (and even λ-Borel) images and preimages, and also under unions and intersections of size λ. In particular, λ-$\mathbf{Bor} \subseteq \lambda$-$\mathbf{\Delta}^1_1$.

Proposition 2

Let $A, B \subseteq X$ be disjoint λ-analytic sets. Then A and B are separated by a λ-Borel set. In particular, λ-$\mathbf{\Delta}^1_1 \subseteq \lambda$-$\mathbf{Bor}$.

Corollary

Assume $2^{<\lambda} = \lambda$. Then λ-$\mathbf{\Delta}^1_1 = \lambda$-$\mathbf{Bor}$.

Generalized Lusin-Suslin theorem

Crucial remark

- For Proposition 1, and in particular to show that λ-$\boldsymbol{\Sigma}^1_1$ is closed under λ-sized intersections, one crucially works with the generalized Cantor space ${}^\lambda 2$; thus such result escapes Approach 1.
- For Proposition 2, we work instead with $B(\lambda)$: it cannot be proved in the realm of Approach 2, unless one moves to cardinals with countable cofinality.

Only the combination of the two approaches (for the right cardinals) can yield the desired result!

Some consequences

The generalized Lusin-Suslin theorem unlocks many useful results and techniques that are out of reach when developing generalized DST as in Approach 1 or Approach 2. For example, assuming $2^{<\lambda} = \lambda$ we have:

1. A function $f \colon X \to Y$ is λ-Borel if and only if its graph is λ-Borel in $X \times Y$.
2. If $f \colon X \to Y$ is λ-Borel, $A \subseteq X$ is λ-Borel, and $f \restriction A$ is injective, then $f(A) \subseteq Y$ is λ-Borel. In particular, the inverse of a λ-Borel injection is still λ-Borel.
3. A set $A \subseteq X$ is λ-Borel if and only if it is a continuous (equivalently: λ-Borel) image of a closed subset of $B(\lambda)$.
4. If $|X| > \lambda$, then X is λ-Borel isomorphic to $B(\lambda)$. In particular, X and Y are λ-Borel isomorphic if and only if $|X| = |Y|$.

One can also prove other separation theorems for λ-analytic sets: this notably includes the (generalized) Novikov's Separation Theorem, i.e. the fact that λ-$\mathbf{\Sigma}^1_1$ has the generalized separation property.

Standard λ-Borel spaces

It is possible to develop a comprehensive theory of standard λ-Borel spaces, among which we find the Effros λ-Borel space $\mathcal{F}(X)$ consisting of all closed subsets of X. Building on this, one gets an analogue of the Kuratowski-Ryll-Nardzewski's Selection Theorem for $\mathcal{F}(X)$.

Theorem

Assume $2^{<\lambda} = \lambda$. Then there is a sequence $(\sigma_\alpha^X)_{\alpha<\lambda}$ of λ-Borel functions $\sigma_\alpha^X : \mathcal{F}(X) \to X$ such that $\{\sigma_\alpha^X(F) \mid \alpha < \lambda\}$ is dense in F for every $F \in \mathcal{F}(X) \setminus \{\emptyset\}$.

This unlocks other familiar results, like:

Theorem

Let E be an equivalence relation on X induced by a continuous action of a λ-Polish group.

- All E-classes are λ-Borel.
- If all E-classes are G_δ, then E is smooth.

λ-Borel uniformizations

A criterion

Let $\emptyset \neq A \subseteq X \times Y$ be a λ-Borel set. TFAE:

1. A has a λ-Borel uniformization;
2. the topology on $X \times Y$ can be refined to a λ-Polish topology τ with the same λ-Borel sets such that A is τ-closed and each vertical section A_x has an isolated point.

Generalized Lusin-Novikov Theorem

Assume $2^{<\lambda} = \lambda$. If $A \in \lambda\text{-}\mathbf{Bor}(X \times Y)$ has countable vertical sections, then it has a λ-Borel uniformization; moreover, it can be covered by countably many pairwise disjoint λ-Borel graphs of partial functions.

Warning!

The proof is radically different from the classical one, as we lack Baire category methods. It crucially uses the fact that $\lambda > 2^{\aleph_0}$.

Generalized Feldman-Moore Theorem

Theorem

Assume $2^{<\lambda} = \lambda$, and let E be an equivalence relation on a standard λ-Borel space X. TFAE:

1. E is λ-Borel and all its classes are countable;
2. E is induced by a countable group acting on X by λ-Borel automorphisms.

Similar results can be proved for equivalence relations with compact classes. More recent (still unpublished) work aims at dealing with λ-Borel equivalence relations with classes of size at most λ.

Other topics that have already been considered include the following:

- ranks for λ-coanalytic sets;
- reflection theorems;
- ...

λ-Perfect Set Property

Definition

Let X be a λ-Polish space. A set $A \subseteq X$ has the λ-**Perfect Set Property** (λ-PSP) if either $|A| \leq \lambda$, or ${}^{\lambda}2$ embeds into A as a closed-in-X set.

Given that ${}^{\lambda}2$ is neither compact nor λ-Lindelöf, this might seem too strong. However, under $2^{<\lambda} = \lambda$ the λ-PSP is equivalent to: Either $|A| \leq \lambda$, or there is a λ-Borel injection $f \colon {}^{\lambda}2 \to A$.

As in the classical case:

- All λ-analytic sets have the λ-PSP.
- Under AC, there is a set without the λ-PSP.
- Assume that $\mathsf{V} = \mathsf{L}$ (or even just that $0^{\#}$ does not exist). Then there is a λ-coanalytic subset of ${}^{\lambda}2$ without the λ-PSP.

Warning!

The latter is trickier than expected, as we lack the analogue of Shoenfield trees for λ-coanalytic sets.

More sets with the (λ-)Perfect Set Property

In the classical setting, assuming the existence of sufficiently large cardinals one can prove that all projective sets have the PSP. This can be proved in two steps, exploiting the notion of κ-weakly homogeneously Suslin sets:

1. If A is κ-weakly homogeneously Suslin, then A has the PSP.
2. Under large cardinals, all sets in $L(\mathbb{R})$ are κ-weakly homogeneously Suslin, for a suitable measurable cardinal κ.

It is a fact that κ-weakly homogeneously Suslin-ness is a sort of "super regularity property", which entails the PSP (together with many other regularity properties) and follows the usual pattern:

- If there is a measurable cardinal κ, all analytic (and in fact: all $\mathbf{\Sigma}^1_2$) sets are κ-weakly homogeneously Suslin.
- Consistently, there are low-level projective sets that are not κ-weakly homogeneously Suslin.
- Under sufficiently large cardinal assumptions, all definable sets (e.g. all sets in $L(\mathbb{R})$) are κ-weakly homogeneously Suslin.

Woodin's approach under I0

This is essentially Approach 3.

Assuming I0(λ), Woodin isolated a technical notion, called $\mathbb{U}(j)$-**representability**, which provides a higher analogue of κ-weakly homogeneously Suslin sets. He then tried to use it to get some form of Perfect Set Property for definable subsets of his space $V_{\lambda+1}$.

Lemma (Woodin)

Let $A \subseteq V_{\lambda+1}$ be a $\mathbb{U}(j)$-representable set in $L(V_{\lambda+1})$, where j witnesses I0(λ). If $|A| > \lambda$, then there is a continuous injection $f \colon {}^{\omega}2 \to A$.

This is fairly week, as the classical Cantor space ${}^{\omega}2$ is incomparably smaller than $V_{\lambda+1}$ (or any other space related to it).

More sets with the λ-PSP

Using completely different techniques, Woodin's lemma was later strengthened by some of his students.

Theorem

Assume $\mathsf{I0}(\lambda)$.

- (Shi) Let $A \subseteq V_{\lambda+1}$ be a set in $L(V_{\lambda+1})$ that is definable in $V_{\lambda+1}$, possibly with parameters in $V_{\lambda+1}$. If $|A| > \lambda$, then there is a continuous injection $f \colon C(\lambda) \to A$.
- (Cramer) Let $A \subseteq V_{\lambda+1}$ be any set in $L(V_{\lambda+1})$. If $|A| > \lambda$, then there is a continuous injection $f \colon B(\lambda) \to A$.

These results are, in a sense, unsatisfactory for a descriptive set theorist, as (unlike Woodin's lemma!) they use deep set-theoretic techniques including absoluteness results, inverse limits, and alike.

All of this can happen only in the exotic space $V_{\lambda+1}$.

$\mathbb{U}$-representable sets

Vindicating Woodin's intuition, we introduce the notion of a $\mathbb{U}$-**representable set**, as well as an analogue of Woodin's **Tower Condition**. Crucially, these notions are defined for subsets of *arbitrary λ-Polish spaces*.

It can be noticed that:

- If $\lambda = \omega$, then this notion coincides with κ-weakly homogeneously Suslin sets.
- Under I0(λ), instead, it generalizes Woodin's $\mathbb{U}(j)$-representability.

Using suitable games, and *without assuming* AC, one can prove that:

Theorem

Assume $2^{<\lambda} = \lambda$, and let X be an arbitrary λ-Polish space. If $A \subseteq X$ admits a $\mathbb{U}$-representation with the Tower Condition, then A has the λ-PSP.

Which sets are $\mathbb{U}$-representable?

One can prove that:

- If there is a measurable cardinal $\kappa > \lambda$, then all λ-analytic sets are $\mathbb{U}$-representable (with the Tower Condition).
- (Barrera-Dimonte-Müller) It is consistent with ZFC+"there is a measurable cardinal $\kappa > \lambda$" that there is a λ-coanalytic set $A \subseteq {}^{\lambda}2$ without the λ-PSP: such an A is not $\mathbb{U}$-representable.
- (Cramer) Under I0(λ), every set $A \subseteq V_{\lambda+1}$ in $L(V_{\lambda+1})$ is $\mathbb{U}(j)$-representable (with the Tower Condition), and therefore it is also $\mathbb{U}$-representable.

As a consequence, we get the ultimate result in this direction:

Theorem

Assume I0(λ), and let X be a λ-Polish space in $L(V_{\lambda+1})$. Then all sets $A \subseteq X$ in $L(V_{\lambda+1})$ have the λ-PSP.

The dark side

Despite the unexpected success of the project, there are some very important missing tools:

- **Compactification:** If λ is at least ω-inaccessible, then every λ-Lindelöf λ-Polish space has size at most λ.
- **Borel determinacy:** There are even λ-$\mathbf{\Delta}^0_2$ subsets of $B(\lambda)$ that are not determined.
- **Absoluteness:** We only have λ-$\mathbf{\Sigma}^1_1$-absoluteness, but no analogue of Shoenfield absoluteness.
- **Measures:** Recent work of Agostini, Barrera, and Dimonte confirms that it is not possible to have decent higher analogues of (λ-Borel) measures.
- **Baire category:** Various attempts, ranging from classical meager sets (countable unions of nowhere dense sets) to λ-meager sets (λ-sized unions of nowhere dense sets). The former is the only notion for which the space $B(\lambda)$ is not meager in itself; unfortunately, there are very simple λ-Borel sets which do not have the (classical) Baire property.

Future work

- The difficulty with Baire category can be overcome by considering an alternative, finer topology on $C(\lambda)$ that is induced by the (diagonal) Prikry forcing. The λ-Baire Property with respect to such topology behaves better: for example, the space $C(\lambda)$ is not λ-meager, and all λ-analytic sets have the corresponding λ-Baire Property. The possibility of extending this to more complicated sets is strictly related to other large cardinals such as I1, I2, and supercompact cardinals (see the work of Dimonte-Iannella-Lücke and Dimonte-Thei).
- Using this and a higher analogue of the $\mathbb{G}_0$-dichotomy, one should be able to obtain a generalization of the Silver's Dichotomy Theorem for λ-coanalytic equivalence relations, and of the Burgess' Trichotomy Theorem for λ-analytic equivalence relations.
- What for singular cardinals of uncountable cofinality? A lot can be said, but new methods are required (work in collaboration with Agostini, Chapman, and Pitton).

That's all, folks!

There are plenty of research ideas that could push generalized descriptive set theory forward... The sky is the limit!

Thank you for your attention!

Main reference:

V. Dimonte and L. Motto Ros, *Generalized Descriptive Set Theory at Singular Cardinals of Countable Cofinality*, arXiv:2511.16188 (151 pages)